Survey on Poincaré Conjecture

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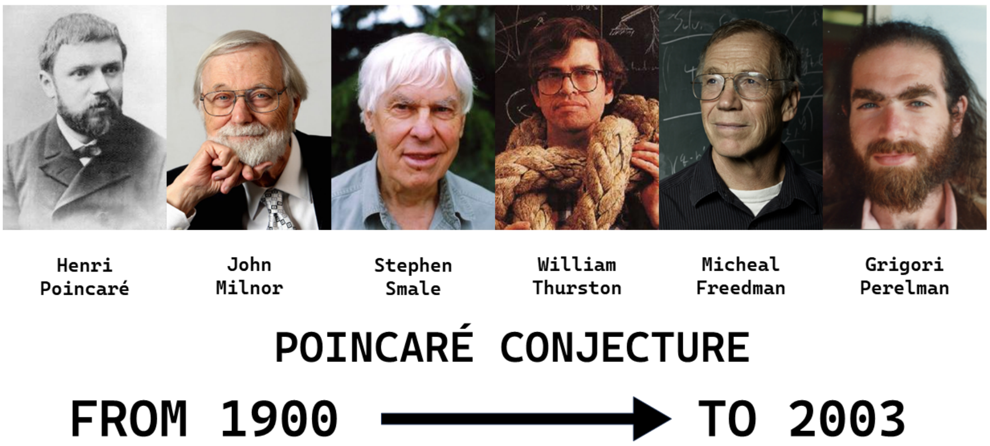

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1 The Poincaré Conjecture

Since its proposal by Henri Poincaré in 1904, the Poincaré Conjecture has driven the flourishing development of geometric topology for over a hundred years. At least five Fields Medals have been awarded in relation to it.

Its research path is similar to that of the Triangulation Conjecture, proceeding by category and dimension. Interestingly, the 3-dimensional triangulation problem was solved early on, whereas the 3-dimensional Poincaré Conjecture was only solved in this century by Perelman; the smooth triangulation problem was proven early, but the smooth Poincaré Conjecture remains unresolved. The smooth Poincaré Conjecture refers to the existence of exotic spheres in different dimensions; in particular, the existence of 4-dimensional exotic spheres is a highly open problem.

1.1 The Poincaré Conjecture

When many concepts of homotopy groups and homology groups in modern topology were first proposed in the early 20th century, their definitions were ambiguous. Stated in modern language, in 1900, Poincaré initially conjectured: if the homology groups of a 3-dimensional manifold are the same as those of the 3-sphere S^3, then it is simply connected, and thus homeomorphic to S^3. However, the first half of this conjecture is wrong. Poincaré provided a counterexample in 1904 (the Poincaré homology sphere P, where $H_*(P) = H_*(S^3)$, but $|\pi_1(P)| = 120$, with the fundamental group being the binary icosahedral group—a perfect group of order 120—so P is not simply connected). However, he retained the second half of his 1900 conjecture, which is:

Conjecture 1 (Poincaré Conjecture (1904))**.** *Let M be a closed 3-dimensional manifold. If M is simply connected, then M is homeomorphic to S^3.*

For a century, this conjecture remained unresolved. Researchers turned to studying the analogue of the Poincaré Conjecture in other dimensions, namely the Generalized Poincaré Conjecture. However, the statement at this point could not simply be "a simply connected closed manifold of dimension $n \geqslant 4$ is homeomorphic to S^n", because it is known that there exist simply connected closed manifolds in these dimensions that are not homotopy equivalent to S^n.

Therefore, for $n \geqslant 4$, we require the condition "homotopy equivalent to S^n", which is stronger than being simply connected. That is:

Conjecture 2 (Generalized Poincaré Conjecture)**.** *Let M be a closed n-dimensional manifold. If M is homotopy equivalent to S^n, then M is homeomorphic to S^n.*

Historically, the term "Generalized Poincaré Conjecture" has often been used indiscriminately to refer to the "Topological Poincaré Conjecture (the one above)", "Weak Smooth Poincaré Conjecture", "Smooth Poincaré Conjecture", "Weak P.L. Poincaré Conjecture", or "P.L. Poincaré Conjecture". In this article, I wish to use it to refer specifically to the "Topological Poincaré Conjecture".

Since a closed 3-dimensional manifold is simply connected if and only if it is homotopy equivalent to S^3 (I will provide a short proof at the end of this paper), the Generalized Poincaré Conjecture for $n = 3$ is precisely the original (narrow) Poincaré Conjecture.

Today, the Generalized Poincaré Conjecture (the specific Poincaré Conjecture when $n = 3$) has been proven correct in all dimensions.

1. When $n = 1$, it is correct, because a closed curve is necessarily homeomorphic to S^1;

2. When $n = 2$, it is correct; by the classification theorem of closed surfaces, a simply connected closed surface must be S^2;

3. When $n = 3$, it is correct. In 2003, Perelman used Ricci flow to prove Thurston's Geometrization Conjecture [Per02][Per03b][Per03a] (i.e., any closed 3-manifold can be decomposed along 2-tori into pieces, each of which admits one of eight geometric structures; a simply connected closed manifold can only admit spherical geometry, i.e., S^3), thereby solving the Poincaré Conjecture;

4. When $n = 4$, it is correct. In 1982, Freedman proved the topological h-cobordism theorem for 4-manifolds, then proved the classification theorem for simply connected 4-manifolds, and finally deduced the 4-dimensional Poincaré Conjecture [Fre82];

5. When $n \geqslant 5$, it is correct. In 1962, the h-cobordism theory proposed by Smale provided a proof for $n \geqslant 6$ [Sma62], but the proof for $n = 5$ is generally credited to Newman in 1966 [New66].

Regarding this series of conclusions, the following supplementary explanations should be made:

- In Smale's era, the Generalized Poincaré Conjecture for dimension 5 could not be proven using the smooth h-cobordism theorem. The smooth h-cobordism theory is only correct for $n \geqslant 5$ (dimension of the cobordism), and it can only be used to prove the Poincaré Conjecture for dimension $n + 1$. It was not until Freedman proved the 4-dimensional topological h-cobordism theorem that a proof based on the h-cobordism theorem could be given. Smale's 1962 paper only claimed results for the P.L. Poincaré Conjecture. His method worked for P.L. and Weak P.L. conjectures for $n \geqslant 5$, but not for the topological conjecture at $n = 5$. Nevertheless, people often attribute the "Poincaré Conjecture for $n \geqslant 5$" to Smale and broadly state that Smale solved the Generalized Poincaré Conjecture for $n \geqslant 5$.

- Freedman's 1982 paper lacked many details. It was not until he personally convinced some prominent figures that the academic community accepted his work. However, his toolkit has rarely been used in subsequent research, and fewer and fewer people truly understand his proof. Most just treat his conclusion as a "black box", and some have even begun to question its correctness. Fearing that the proof of this epoch-making result might be "lost", a group of mathematicians, with Freedman's support, began in 2013 to expand his work into a 496-page book, published in 2021 [BKK+21], with the goal of enabling motivated undergraduates to understand it within a semester [Har21].

- In the three papers Perelman published between 2002 and 2003, not a single sentence mentioned "Geometrization Conjecture" or "Poincaré Conjecture," although his techniques had de facto proven them. Later, several groups of mathematicians filled in the details of the proof, among whom John Morgan and Tian Gang expanded it into a book [MT07]. The academic community attributes the proof of the Poincaré Conjecture to Perelman; however, he declined the 2006 Fields Medal and the $1 million Millennium Prize from the Clay Mathematics Institute in 2010, stating that Hamilton, who pioneered Ricci flow, made an equal contribution, and the resolution of the Poincaré Conjecture should not be credited to him alone. In fact, Perelman had already achieved numerous major results in the 90s. In 1994, he was invited to speak at the ICM for his contributions to Alexandrov geometry and proved the "Soul Conjecture" the same year. In 1995, he rejected faculty offers from several top universities, and in 1996, he rejected a prize from the European Mathematical Society, establishing a long "track record" of declining awards.

Since the conclusion of the "Generalized Poincaré Conjecture" mentioned above only requires the manifold to be homeomorphic to S^n in the topological sense, to distinguish it from the "more generalized" Poincaré conjectures discussed later, we will hereafter refer to the aforementioned Poincaré Conjecture as the "Topological Poincaré Conjecture."

1.2 Smooth Poincaré Conjecture

When the topological Poincaré Conjecture for $n \geqslant 4$ was not yet solved, mathematicians tried to impose stronger regularity on the original manifold, such as smoothness or piece-wise linear (P.L.) structures, to see if conditions for homeomorphism to a sphere could be obtained.

Conjecture 3 (Weak Smooth Poincaré Conjecture)**.** *Let M be a closed n-dimensional smooth manifold. If M is homotopy equivalent to S^n, then M is homeomorphic to S^n.*

Historically, the proof of the Weak Smooth Poincaré Conjecture for $n \geqslant 5$ was given by Smale in his famous 1960 paper "GPC", using tools from differential topology [Sma61]. Almost simultaneously, Stallings provided a proof for the Weak P.L. Poincaré Conjecture for $n \geqslant 7$ [Sta60], which implies the corresponding Weak Smooth Poincaré Conjecture.

However, since the Topological Poincaré Conjecture is stronger than the Weak Smooth Poincaré Conjecture, people seem to care little about the history of the weak smooth conjecture after the topological one was solved.

A more worthy problem to study is to make a trade-off: imposing a stronger condition while also strengthening the conclusion, requiring not just topological homeomorphism but diffeomorphism. We call this balanced conjecture the Strong Smooth Poincaré Conjecture, or simply the Smooth Poincaré Conjecture. It is famous and remains not fully resolved to this day.

Conjecture 4 (Smooth Poincaré Conjecture)**.** *Let M be a closed n-dimensional smooth manifold. If M is homotopy equivalent to S^n, then M is diffeomorphic to S^n equipped with the standard smooth structure.*

For the unit sphere S^n, removing the south and north poles gives $S^n \backslash (\text{S.pt.})$ and $S^n \backslash (\text{N.pt.})$. Taking stereographic projections φ_1 and φ_2 to the equatorial plane R^n, $\mathcal{M} = \{(S^n \backslash (\text{S.pt.}), \varphi_1), (S^n \backslash (\text{N.pt.}), \varphi_2)\}$ is a smooth atlas for S^n. The "standard smooth structure" of S^n mentioned above is the maximal atlas compatible with $\mathcal{M}$.

In fact, conclusions regarding the Smooth Poincaré Conjecture were proposed long before the series of developments in the Topological Poincaré Conjecture.

Theorem 1 (Moise[Moi52]+[Mil11]Theorem2)**.** *When $n \leqslant 3$, n-dimensional topological manifolds have a unique smooth structure.*

This theorem tells us that when $n \leqslant 3$, topological homeomorphism is equivalent to smooth diffeomorphism, meaning the Topological Poincaré Conjecture for $n \leqslant 3$ is equivalent to the Smooth Poincaré Conjecture.

Theorem 2 (Milnor[Mil56])**.** *Exotic smooth structures exist on S^7.*

This conclusion disproved the Smooth Poincaré Conjecture for $n = 7$. If we denote a sphere with an exotic smooth structure as Σ^7, then Σ^7 is homeomorphic (and thus homotopy equivalent) to S^7 but not diffeomorphic to it. We refer to such a Σ^7 as an "exotic sphere".

Following the publication of Milnor's 1956 result, research on exotic smooth structures on smooth manifolds flourished. Research on "exotic spheres" is essentially research on the "Smooth Poincaré Conjecture": if S^n possesses a smooth structure not diffeomorphic to the standard one, then the n-dimensional Smooth Poincaré Conjecture is false. Since the Topological Poincaré Conjecture is correct in all dimensions, we can reduce the statement of the Smooth Poincaré Conjecture to a more direct version:

Conjecture 5 (Alternative Statement of the Smooth Poincaré Conjecture)**.** *Let M be a closed n-dimensional smooth manifold. If M is homeomorphic to S^n, then M is diffeomorphic to the standard differential structure of S^n; that is, there are no n-dimensional exotic spheres.*

For authoritative reviews on the Smooth Poincaré Conjecture, one can refer to the introduction of Guozhen Wang's article [WX17], as well as the lecture slides by Zhouli Xu in November 2024 [Xu24].

After Milnor proposed the existence of exotic structures on S^7 in 1956, he and Kervaire discussed the 7-dimensional case completely in 1963 [KM63]: there are 28 (considering orientation) smooth structures on S^7 that are not diffeomorphic; if orientation is not considered, there are 15.

In that paper, Kervaire and Milnor provided a general method for determining the existence of exotic spheres: calculating the group of homotopy n-spheres Θ_n. From Smale's smooth h-cobordism theorem for $n \geqslant 5$, it is known that for $n \geqslant 5$, the number of distinct smooth structures on S^n is simply $|\Theta_n|$. They also gave conclusions for dimensions 5 through 18, noting that dimensions 5, 6, and 12 do not have exotic spheres.

Today, the theory of exotic spheres has the following conclusions:

In sufficiently high dimensions, all odd-dimensional spheres possess exotic smooth structures. Specifically, the only odd-dimensional spheres with a unique smooth structure are S^1, S^3, S^5, and S^{61}. The fact that the last odd dimension, 61, has no exotic spheres was proven by Guozhen Wang and Zhouli Xu in 2017 [WX17].

More than half of the even dimensions have been proven to possess exotic spheres; it is conjectured that they exist in the remaining even dimensions as well [BMQ23].

Current Conjecture:

Conjecture 6. *For spheres of dimension greater than 4, the only ones with a unique smooth structure are S^5, S^6, S^{12}, S^{56}, and S^{61}.*

Progress in the theory of exotic spheres is currently very rapid, and people believe this conjecture is correct.

Regarding the existence of 4-dimensional exotic spheres, although it is a highly open problem, people tend to believe that 4-dimensional exotic spheres do exist. This is because 4-dimensional space possesses too many "wild" properties: for example, $\mathbb{R}^4$ has uncountably many smooth structures that are not mutually diffeomorphic (though this conclusion does not currently contribute to the existence of 4-dimensional exotic spheres).

1.3 P.L. Poincaré Conjecture

Similar to the Smooth Poincaré Conjecture, the P.L. (Piecewise Linear) Poincaré Conjecture also initially had a weak version:

Conjecture 7 (Weak P.L. Poincaré Conjecture)**.** *Let M be a closed n-dimensional P.L. manifold. If M is homotopy equivalent to S^n, then M is homeomorphic to S^n.*

Historically, the "Weak P.L. Poincaré Conjecture" developed as follows:

1. In 1960, Smale [Sma61] and Stallings [Sta60] independently proved the case for $n \geqslant 7$ (Smale announced it first; the two used different methods, with Smale not yet using h-cobordism at this time, and Stallings using a method called "engulfing"). Subsequently, Smale generalized his proof method to $n \geqslant 5$;

2. In 1961, Zeeman modified Stallings' construction [Zee61] and solved the cases for $n = 5, 6$;

3. In 1966, Newman generalized Stallings' engulfing method to the topological case, proving the Generalized (Topological) Poincaré Conjecture for $n \geqslant 5$ [New66].

Smale's initial proof and Stallings' engulfing theorem have a strong flavor of P.L. topology.

P.L. topology had extensive applications in the 1960s and 70s, through which Smale and others achieved a host of brilliant results. However, with the resolution of a series of major problems, this set of tools gradually "declined". Today, the popular tool in geometric topology is Gauge Theory, and few people use P.L. topology tools to solve problems anymore. Regarding a series of methods and applications of P.L. topology, one can refer to a long review written by Sandro Buoncristiano in 2003 [Buo03], which introduces the details of the proof of the Weak P.L. Conjecture; and the textbook written by Rourke and Sanderson in the 80s [RS82], which introduces the proof of the P.L. conjecture for $n \geqslant 6$ based on the h-cobordism theorem.

With the resolution of the Topological Conjecture, the Weak P.L. Conjecture remains mostly of technical value. Nowadays, when introducing the proof methods of the Generalized Poincaré Conjecture, people often use the h-cobordism theorem, even though historically it was not the h-cobordism theorem that first proved the Generalized Poincaré Conjecture.

The strong version of the Weak P.L. Poincaré Conjecture after trade-off (usually directly called the P.L. Poincaré Conjecture) is:

Conjecture 8 (P.L. Poincaré Conjecture)**.** *Let M be a closed n-dimensional P.L. manifold. If M is homotopy equivalent to S^n, then M is P.L. homeomorphic to S^n.*

The P.L. structure is a weaker structure than the smooth structure. One conclusion is that every smooth structure determines a unique P.L. structure [Cai35][Whi40], and P.L. regularity is stronger than topological. By Theorem 1, when $n \leqslant 3$, the Topological, P.L., and Smooth categories are equivalent. Since the Topological Poincaré Conjecture is correct for $n \leqslant 3$, the P.L. Poincaré Conjecture for $n \leqslant 3$ is also correct.

The P.L. Poincaré Conjecture for $n \geqslant 5$ was solved by Smale using methods related to the h-cobordism theorem in 1962 [Sma62].

Since smooth structures are equivalent to P.L. structures for $n \leqslant 6$ [Mil11] (Theorem 2), the 4-dimensional P.L. Poincaré Conjecture is equivalent to the existence of 4-dimensional exotic spheres.

That is:

Corollary 1.

$$\begin{aligned} &\text{4-dimensional P.L. Poincaré Conjecture is correct} \\ \iff &\text{4-dimensional Smooth Poincaré Conjecture is correct} \\ \iff &S^4 \text{ has no exotic smooth structures} \end{aligned}$$

1.4 Generalized Conjecture for $n = 3$ is Equivalent to the Narrow Conjecture

Proposition 1. *Let M^3 be a closed 3-dimensional manifold. Then M is simply connected $\iff$ M is homotopy equivalent to S^3.*

Proof. $\Leftarrow$: $\pi_1(M) = \pi_1(S^3) = 0$, so M is simply connected;

$\Rightarrow$: If M is simply connected, its connected orientable covering is the trivial covering, meaning M is an orientable manifold, so $H_3(M) = \mathbb{Z}$. Additionally, since $H_1(M)$ is the abelianization of $\pi_1(M)$, $\pi_1(M) = 0$ implies $H_1(M) = 0$. By the Universal Coefficient Theorem, $H^1(M) = 0$. Then by Poincaré Duality, $H_2(M) \cong H^1(M) = 0$. By the Hurewicz Theorem, $\pi_2(M) \cong H_2(M) = 0$, and consequently $\pi_3(M) \cong H_3(M) \cong \mathbb{Z}$. This means a generator of $\pi_3(M)$ can be determined by a map $S^3 \to M$ of degree 1, inducing an isomorphism between H_3 and π_3. Furthermore, there exists a map from S^3 to M (regarded as simply connected simplicial complexes) that induces isomorphisms on all homology groups. By Whitehead's Theorem, this map is a homotopy equivalence. □

The idea for the necessity part of the proposition comes from Hatcher's review article on the classification of 3-manifolds [Hat04], and one can also refer to a Zhihu article [梁 19].

2 Summary

Essentially, the only remaining unresolved part of the Poincaré Conjecture involves exotic spheres in certain even dimensions (especially 4).

Table 1: Status of Resolution for Various Poincaré Conjectures

Category	Status
Topological	Correct in all dimensions
P.L.	Correct in all dimensions except $n = 4$ (unresolved)
Smooth	Correct for $n = 1, 2, 3, 5, 6, 12, 56, 61$,
	Incorrect for all other odd dimensions, incorrect for > half of even dimensions,
	Unresolved for less than half of even dimensions (especially 4)

To summarize the five Fields Medals associated with the Poincaré Conjecture, they were awarded sequentially to Milnor, Smale, Thurston, Freedman, and Perelman.

Table 2: Fields Medals Related to the Poincaré Conjecture

Mathematician	Year Awarded	Achievement
John Milnor	1962	7-dimensional exotic sphere (7D Smooth PC)
Stephen Smale	1966	Generalized (P.L.) Poincaré Conjecture for $n \geqslant 5$
William Thurston	1982	Geometrization Conjecture (Series of results on 3-manifolds)
Michael Freedman	1986	Generalized Poincaré Conjecture for $n = 4$
Grigori Perelman	2006	(Specific) Poincaré Conjecture

Based on this statistical pattern, can we infer that the 2026 Fields Medal will also be awarded for the Poincaré Conjecture? If we discount the steadily progressing problem of high-dimensional exotic spheres, only the large puzzle piece of the 4-dimensional exotic sphere remains for the Poincaré Conjecture. Time is running out for the mathematics community :).

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