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流形的双曲化与不可三角剖分的流形

Hyperbolization versus Non-triangulable Manifolds

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流形的双曲化与不可三角剖分的流形

摘　要

Gromov 在 1980 年代提出了双曲化的方法，这种方法能将任意一个多面体转化成非正曲率的空间，并保持原有的局部结构。Michael W. Davis 等人之后又进一步发展了这种方法。双曲化可以广泛地用于流形上，所以某种意义上可以认为“双曲流形在所有流形中是稠密的”。然而“不可三角剖分”的流形是一类例外，它们无法直接用以 Gromov 的双曲化方法。不过，我们能够折衷在 4 维和大于等于 6 维把一些不可三角剖分的流形“非球面化”，这是比双曲化更弱的要求。然而，现在仍没有构造出 5 维的非球面化的不可三角剖分的流形。不可三角剖分的流形上是否存在 CAT(0) 度量也仍是一个开放问题。

关键词：双曲化，三角剖分，三角剖分猜想，非球面化，CAT(0) 空间

HYPERBOLIZATION VERSUS NON-TRIANGULABLE MANIFOLDS

ABSTRACT

Gromov proposed hyperbolization in 1980s as a technique to transform a polyhedron into a non-positively curved space while preserving the local structure, which has been develop by Michael W. Davis et al. This technique can be applied to a wide class of manifold, so hyperbolic manifolds are somehow dense in the space of manifolds. However, the non-triangulable manifolds are exception. If we trade off to do asphericalization instead of hyperbolization, we can have some examples of aspherical non-triangulable manifolds in dimension 4 and greater than 5. But we still don't have examples in dimension 5. And whether a non-triangulable manifold admits locally CAT(0) metric or not remains open.

KEY WORDS: Hyperbolization, Triangulation, Triangulation Conjecture, Asphericalization, CAT(0) space

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Chapter 1 Introduction

Hyperbolization is a technique to canonically transform polyhedra, a kind of well-behaved topological space, into non-positively curved spaces, in other word, locally CAT(0) spaces, while it preserves the local structure of the original spaces. It's proposed by Gromov in the 1980s [10], and elaborated by Davis and Januszkiewicz in the 1990s [6]. This technique then have evolved in several decades: Charney and Davis achieved strict hyperbolization, that is, to transform polyhedra into locally CAT(-1) spaces in 1995 [4], and Ontateda in the 2010s achieved "Riemannian hyperbolization" [18].

Since the hyperbolization technique can be widely used for polyhedra, how about manifolds, the another kind of well-behaved topological space? Then it encounters a famous problem in Geometric Topology, the Triangulation Conjecture, stating that "All topological manifolds admit triangulations, i.e. they are also polyhedra." proposed by Kneser in the 1920s. If the conjecture is true, then the standard hyperbolization offers a shortcut to transform manifolds into hyperbolic manifolds. Or at least if non-triangulable manifolds are rare, then we can claim that "Hyperbolic manifolds are somehow dense in the space of all manifolds".

Good news is that the Triangulation Conjecture holds for all smooth manifolds [3][22] or in dimension $\leqslant 3$ [17]. However, in general case it is wrong. The first counterexample occurred in dimension 4. In Freedman's groundbreaking work for 4-manifold in 1982 [7], the E_8 manifold M_{E_8} was constructed and proved to admit no smooth structure. And later in 1990, Casson proved that it also admits no triangulation [1]. Unfortunately, for dimension $\geqslant 5$, the conjecture is also wrong. Galewski and Stern in the 1970s constructed a "universal 5-manifold" N^5, possessing the property that if it is triangulable, then all manifolds in dimension $\geqslant 5$ are triangulable [8]. And Manulescu in the 2010s proved that their exists non-triangulable manifolds in dimension $\geqslant 5$ based on Galewski and Stern's work. So N^5 turns out to be an example, and the same is true for $N^5 \times T^{n-5}$ in all dimension $n \geqslant 5$.

It is still open that whether there exists a way to hyperbolize general manifolds, or there exists a non-triangulable manifold admitting no locally CAT(0) metric. Nevertheless, there has been progress under a weaker assumption on non-triangulable manifold as a trade-off, that is, asphericity instead of non-positively curved metric. For non-positively curved space, the generalized Cartan-Hadamard theorem (similar to the classical version in Riemannian geometry) implies that its universal cover is contractible, hence the space itself is aspherical. Davis and Januszkiewicz provided an example that is a aspherical non-triangulable manifold in dimension 4, based on the construction of M_{E_8} [6], and Davis, Fowler and Lafont in 2013 provided examples in dimension $\geqslant 6$ [5] based on the construction of the “universal 5-manifold”. However, the problem still remains open in dimension 5.

This paper is organized as follows. In Chapter 2 we introduce the hyperbolization technique in detail, following the construction given in [6]. In Chapter 3 we introduce the development of the Triangulation Conjecture, and show how the non-triangulable manifolds are constructed in dimension 4 and $\geqslant 5$. In Chapter 4, we combine the arguments in the previous chapters, and show how to construct aspherical non-triangulable manifolds in dimension 4 and $\geqslant 6$. At last, we analyze the difficulties to hyperbolize a non-triangulable manifold.

Chapter 2 Hyperbolization of Polyhedra

A polyhedron is a topological space with a simplicial complex structure. We usually denote the polyhedron by $|K|$, if its simplicial complex is K. Hyperbolization is a process to transform a polyhedron into a metric space with non-positive curvature in the sense of Gromov. It's described in §3.4 of [10], and elaborated by Davis and Januszkiewicz [6].

Intuitively, this procedure is to replace the simplices of a polyhedron $|K|$ into consistent hyperbolic blocks.

This construction makes sense because the "non-positively curved" property preserves when gluing two pieces along the boundary. Thus we could reproduce a new space following the pattern of how simplices assembled in the original polyhedron, and using the "hyperbolized simplices" X instead. Then the new space denoted by $X \Delta K$ shares the same local structure with K, and turns out to be non-positively curved, which implies asphericity.

Theorem 2.1 (Davis & Januszkiewicz [6])**.** *Suppose that (X^n, f) is a hyperbolized n-simplex which is degree one and tangentially trivial. For any n-dimensional simplicial complex K, let $a(K) := X\Delta K$. Then $a(K)$ (with intrinsic metric) is a non-positively curved geodesic space, hence an aspherical space.*

Here (X, f) serves as fundamental "block". In this chapter, we will explain in detail how this technique works.

2.1 Williams Functor

Now we introduce the key step of constructing hypebolized polyhedron by means of "fiber production", rigorously showing how to build up a new space using hyperbolic blocks with the pattern of a simplicial complex. This switch from one simplicial complex into a new space is called Williams functor [23].

2.1.1 Basic Settings

Definition 2.1. Let σ^n be the standard n-simplex.

1. A space over σ^n is a pair (X, f), where X is a topological space and $f : X \to \sigma^n$ is a continuous map.
2. A simplicial complex over σ^n is a pair (L, π), where L is a simplicial complex and $\pi : L \to \sigma^n$ is a nondegenerate simplicial map (implying $\dim L \leqslant n$).

Suppose that (X, f) is a space over σ^n and (L, π) is a simplicial complex over σ^n, then we can define the fiber production of X and $|L|$ over σ^n,

$$X\tilde{\Delta}L := \{(x, y) \in X \times |L| \mid f(x) = \pi(y)\},$$

as a subspace of $X \times |L|$.

Furthermore, denote the natural projections as $f_L : X\tilde{\Delta}L \to |L|$ and $p : X\tilde{\Delta}L \to X$, then we have the commutative diagram

$$\begin{array}{ccc} X\tilde{\Delta}L & \xrightarrow{f_L} & |L| \\ \downarrow p & & \downarrow \pi \\ X & \xrightarrow{f} & \sigma^n \end{array}$$

If J is any subset of the standard simplex σ^n, denote $X_J := f^{-1}(J)$. In particular, if α is a closed face of σ^n, then X_α is called a face of X.

Remark 2.1. The translation $(L, \pi) \rightsquigarrow (X\tilde{\Delta}L)$ is actually a functor, known as **Williams functor**. This functor is from the category of simplicial complexes over σ^n to the category of topological spaces.

Besides, $(X, f) \rightsquigarrow (X\tilde{\Delta}L)$ from the category of spaces over σ^n and face-preserving maps to the category of topological spaces is also functorial.

Intuitively, the non-degeneracy of (L, π) means that L can be "folded" by several times into one single σ^n. For example the octahedron (with $8 = 2^3$ faces) can naturally be folded by 3 times, each reducing half of faces, and into one triangle face.

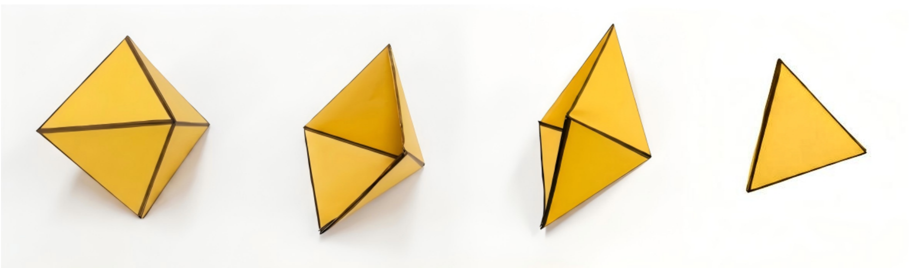


Figure 1 Octahedron folded into a triangle.

Therefore each adjacent pair of facets in L corresponds to two pieces of X in $X\tilde{\Delta}L$ with the intersecting part as the plane of symmetry, obeying the rule of "folding process".

Example 2.1. Let L be a sphere triangulated as octahedron, and π be the natural simplicial projection shown in Figure 1.

1. (Torus.) X is a torus deleted a disk, and $f|_{\partial X \cong S^1}$ is a homeomorphism to $\partial\sigma^2$. Then $X\tilde{\Delta}L$ is a octahedron with each face replaced by "torus piece", which equivalent to a ball added 8 1-handle, i.e. a genus 8 surface.
2. (Flat ring.) $X = \partial\sigma^2 \times I$, and f restricts to the identity on each component of ∂X. $X\tilde{\Delta}L$ is octahedron with each face replaced by "flat ring piece", intuitively (the boundary of) a thickened octahedron with each face punched a large hole.

3. (2-fold branched.) X is a hexagon and $f : X \to \sigma^2$ is a 2-fold branched cover with the center of σ^2 as branch point. We can forcibly realize this 2-dimensional object in 3-dimensional space, but it will inevitably produce self-intersection lines. Although $X\tilde{\Delta}L$ is an orientable closed surface, it's very hard to count genus directly. But we can tell it from it's Euler characteristic: it's glued up by 8 hexagon, and each edge is shared by 2 faces and each vertex shared by 4 faces, thus $V - E + F = 12 - 24 + 8 = -4$. That is to say, $X\tilde{\Delta}L$ is a genus 3 surface.

The three (X, f)'s are demonstrated in Figure 2. The orange boundary of X are mapped to the yellow boundary of σ^2, and points elsewhere are "flattened" onto the interior of σ^2.

See Figure 3 and 4 for more visualization.

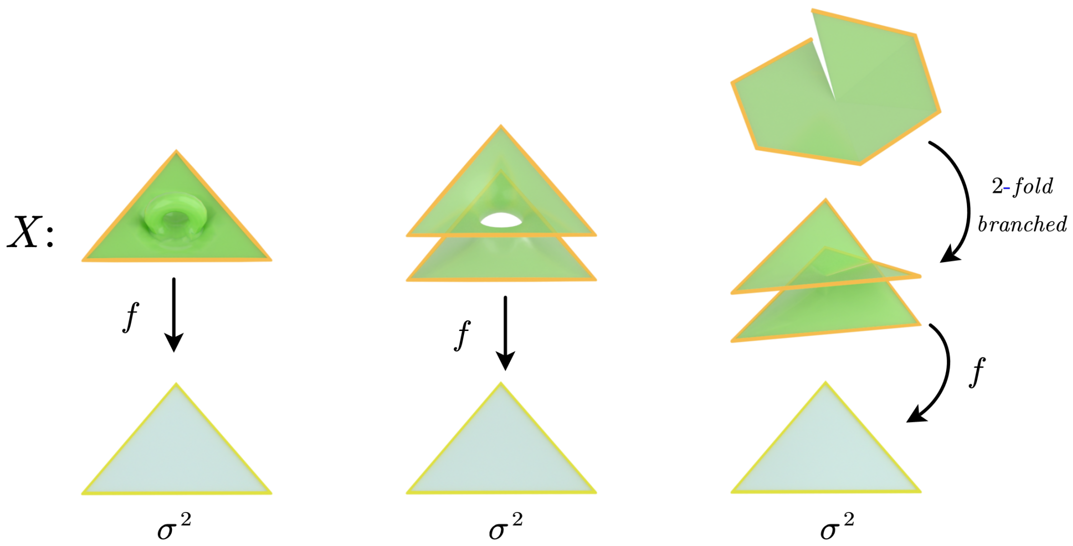


Figure 2 Three examples of (X, f) over σ^2.

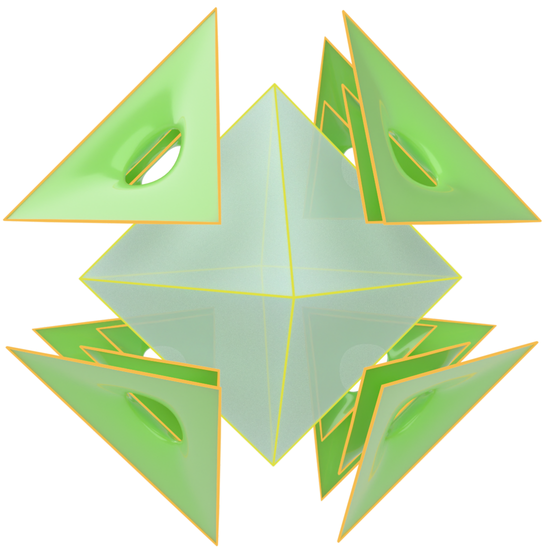
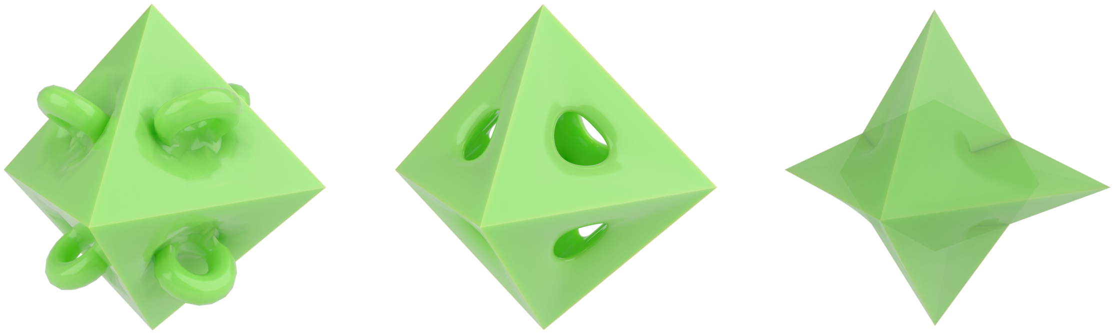


Figure 3 How pieces of X assembled according to L.



(a) By torus pieces. (b) By flat ring pieces. (c) By 2-fold branched pieces.

Figure 4 The resulting $X\tilde{\Delta}L$'s.

Remark 2.2. For the above 3 cases, the degree of f (defined from the map $f_* : H_2(X, \partial X) \to H_2(\sigma^2, \partial\sigma^2)$) is 1, 0, 2, respectively.

Example 2.2 (Natural simplicial complex over σ^n)**.** Let K be an arbitrary simplicial complex, and K' be its derived complex (to barycenter refine each face of K). Let

$$d : K \to \{0, 1, \dots, n\}$$

be the function which assigns to each simplex its dimension. As a simplicial complex, faces of each simplex $\sigma^n \in K$ can be identified with the poset of nonempty subsets of $\{0, 1, \dots, n\}$, thus d naturally induces a non-degenerate simplicial map $K' \to \sigma^n$. Then (K', d) over σ^n is called natural simplicial complex of K.

For K and any space (X, f) over σ^n, we can construct

$$X\Delta K := X\tilde{\Delta}K'$$

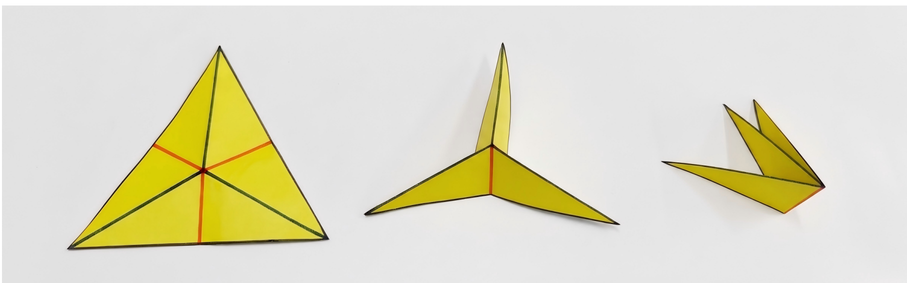


Figure 5 A barycenter refined triangle folded into one piece of triangle.

To have more good properties of $X\tilde{\Delta}L$, we need some requirements on (X, f) to be "nice". In advance, we list some commonly used conditions for (X, f):

(C0) X is path connected and for each codimension one face α of σ^n, the face X_α is nonempty.

(C1) X is a compact n-dimensional P.L. manifold with boundary. Moreover, for each k-dimensional face α of σ^n, X_α is a k-dimensional P.L. submanifold of ∂X, and $\partial(X_\alpha) = X_{\partial\alpha}$. The map $f : X \to \sigma^n$ is also required to be piecewise linear.

(C1') X is a compact smooth n-dimensional manifold with corners. Moreover, for each k-dimensional face α of σ^n, X_α is a union of k-dimensional strata. The map $f : X \to \sigma^n$ is required to be smooth and transverse to each proper face of σ^n.

(C2) X satisfies (C1), and $f : (X, \partial X) \to (\sigma^n, \partial\sigma^n)$ is degree one mod 2.

(C2') X satisfies (C1) and oriented, and $f : (X, \partial X) \to (\sigma^n, \partial\sigma^n)$ is degree one.

(C3) X satisfies (C1) and τ_X is trivial.

(C4) X is an aspherical CW-complex and if P is any subcomplex of σ^n, then each component of X_P is aspherical and the inclusion $i : X_P \to X$ induces a injection $i_* : \pi_1(X_P, x_0) \to \pi_1(X, x_0)$.

(C5) X is a geodesic space of curvature $\leqslant 0$. Moreover, for each connected subcomplex J of σ^n, the subspace X_J is totally geodesic. (Condition (C5) implies condition (C4) by Generalized Cartan-Hadamard Theorem 2.5).

Here the conditions (C4) and (C5) will be used for asphericalization and hyperbolization respectively. We will give definitions of these concepts later.

The following lemma shows the similarity of K and $X\tilde{\Delta}K$ if (C0) or (C2) is permitted.

Lemma 2.1. *Suppose that (X, f) satisfies (C0), then $X\tilde{\Delta}L$ is path connected, and $(f_L)_* : \pi_1(X\tilde{\Delta}L) \to \pi_1(L)$ is surjective.*

This is because they share essentially the same 1-skeleton, which is homotopy equivalent to a wedge sum of circles.

Lemma 2.2 (Williams [23, 2.4, p. 323])**.**

1. *If (X, f) satisfies (C2), then the map $(f_L)_* : H_*(X\tilde{\Delta}L; \mathbb{Z}/2) \to H_*(L; \mathbb{Z}/2)$ is surjective.*
2. *If (X, f) satisfies (C2'), then $(f_L)_* : H_*(X\tilde{\Delta}L; f_L^* A) \to H_*(L; A)$ is surjective, where A is any local coefficient system on $|L|$.*

Proof. Firstly we assume (C2'). Define a chain map $j : C_*(L) \to C_*(X\tilde{\Delta}L)$ by sending a k-simplex γ to the k-chain $\langle X\tilde{\Delta}\gamma\rangle \in C_k(X\tilde{\Delta}\gamma) \subset C_k(X\tilde{\Delta}L)$. Since $f : (X, \partial X) \to (\sigma^n, \partial\sigma^n)$ is degree one and satisfies (C1), $f|_{X_\alpha} : (X_\alpha, \partial X_\alpha) \to (\alpha, \partial\alpha)$ is also degree one, for any k-face α of σ^n. In this way, let $(f_L)_\# : C_*(X\tilde{\Delta}L) \to C_*(L)$ be the chain map induced by f, we have $(f_L)_\#(X\tilde{\Delta}\gamma) = \gamma + \partial\beta$, because of degree one in the homology ($X\tilde{\Delta}\gamma$ can be identified with X_γ). So consider $((f_L)_\# \circ j)(\gamma) = \gamma + \partial\beta$, we have that $(f_L)_* \circ j_* = \mathrm{id} : H_*(L; A) \to H_*(L; A)$. So $(f_L)_*$ is surjective, which completes the proof of the second part of the lemma.

For condition (C2), the proof of the first part is the same when we change to $\mathbb{Z}/2$ coefficient. $\square$

2.1.2 Comparisons Between $X\tilde{\Delta}L$ and X

For any simplicial complex L, the spaces $X\tilde{\Delta}L$ and L have similar local structures, i.e., they have isomorphic links (so as $X\Delta K$ and K). Furthermore, their tangent bundles usually have pullback relationship.

Suppose that α is a k-simplex of K, the **link** of α in K is denoted by

$$\mathrm{Link}(\alpha, K) := \{\beta \in K \big| \alpha * \beta = n\text{-simplex} \in K\},$$

and the following theorem says that $X\tilde{\Delta}L$ and L "shares the same link".

Theorem 2.2 ([6, p. 355-356])**.** *Suppose L is a simplicial complex and (X, f) satisfies (C1) or (C1'). For all simplex α in L, the link of α is isomorphic to the link of X_α in $X\tilde{\Delta}L$.*

Define the dual cone of α in K by

$$\mathrm{Dual}(\alpha, K) := \mathrm{Cone}(\mathrm{Link}(\alpha, K)).$$

Further the open dual cone $\mathrm{Dual}^\circ(\alpha, K) := \mathrm{Dual}(\alpha, K) - \mathrm{Link}(\alpha, K)$.

By definition, the link of α is $n-k-1$ dimensional object, and dual cone is $n-k$ dimensional object. For any σ^n contains α, $\sigma^n \cap \mathrm{Link}(\alpha, K)$ is a $n-k-1$-dimensional simplex, we denote it by β'. Then there is a natural isomorphism $\mathring{\sigma}^n \cong \mathring{\alpha} \times (\mathring{C}\beta)$, where $(\mathring{C}\beta)$ contains in $\mathrm{Dual}^\circ(\alpha, K)$. Therefore, $\mathring{\alpha} \times \mathrm{Dual}^\circ(\alpha, K)$ is well characterized the local structure of L. And the case $X_{\mathring{\alpha}} \times \mathrm{Dual}^\circ(\alpha, L)$ for $X\tilde{\Delta}L$ is similar. This argument concludes the following lemma:

Lemma 2.3. *Suppose that (X, f) is a space over σ^n satisfying (C1) or (C1'), L is a simplicial complex over σ^n, and that α is a k-simplex in L.*

1. *$X_{\mathring{\alpha}}$ has a product bundle open neighborhood in $X\tilde{\Delta}L$ of the form $X_{\mathring{\alpha}} \times \mathrm{Dual}^\circ(\alpha, L)$.*
2. *The map $f_L : X\tilde{\Delta}L \to L$ induces a bundle map $X_{\mathring{\alpha}} \times \mathrm{Dual}^\circ(\alpha, L) \to \mathring{\alpha} \times \mathrm{Dual}^\circ(\alpha, L)$ from a neighborhood of $X_{\mathring{\alpha}}$ in $X\tilde{\Delta}L$ to a neighborhood of $\mathring{\alpha}$ in L. Moreover, this map has the form $g \times \mathrm{id}$, where $g = f_L|_{X_{\mathring{\alpha}}}$.*

Since the links are the "boundary" of dual cones, the links of X_α and α are isomorphic by the isomorphism of their dual complexes, which derives Theorem 2.2.

A n dimensional **P.L. manifold** is defined to be a manifold with transition maps are piecewise linear, which equivalent to require that the manifold $|K|$ with simplicial complex structure K satisfies each link of k-simplex is P.L. homeomorphic to standard $(n-k-1)$ sphere. A n dimensional **homology manifold** requires also a simplicial complex structure, whose homology of links is isomorphic to spheres'.

The following corollary is directly derived by Theorem 2.2.

Corollary 2.1. *Suppose X satisfies (C1) or (C1').*

1. *If $|L|$ is homology n-manifold, then so is $X\tilde{\Delta}L$.*
2. *If $|L|$ is a P.L. n-manifold, then so is $X\tilde{\Delta}L$.*

Next we introduce some tangential properties of $X\tilde{\Delta}L$ [6, p. 356-358].

Proposition 2.1. *Suppose that X satisfies (C1) and that L is a P.L. n-manifold. Then $X\tilde{\Delta}L$ is an n-dimensional P.L. submanifold of $X \times |L|$, and the normal bundle of $X\tilde{\Delta}L$ in $X \times |L|$ is trivial.*

Proof. Denote P^n to be the image of $\sigma^n \times \sigma^n$ under the map $\sigma^n \times \sigma^n \to \mathbb{R}^n$ given by $(u, v) \to u - v$. So P^n is a convex polyhedron and contains the origin in its interior.

Let $\varphi : X \times |L| \to P^n$ be defined by $\varphi(x, y) = f(x) - \pi(y)$, thus

$$X\tilde{\Delta}L = \varphi^{-1}(0).$$

Since φ is piecewise linear and transverse to 0, it derives that $X\tilde{\Delta}L$ is an n-dimensional P.L. submanifold of $X \times |L|$ with trivial normal bundle. $\square$

Corollary 2.2. *Suppose that X satisfies (C3) and L is a P.L. n-manifold. Then the stable tangent P.L. block bundle of $X\tilde{\Delta}L$ is the pullback of the stable tangent P.L block bundle of L, i.e.*

$$T(X\tilde{\Delta}L) = (f_L)^* TL.$$

Proof. Since the restriction $T(X \times |L|)$ to $X\tilde{\Delta}L$ is $T(X\tilde{\Delta}L)$ plus the (trivial) normal bundle of $X\tilde{\Delta}L$, $T(X\tilde{\Delta}L)$ is stably equivalent to the restriction of $T(X \times |L|)$. Further, by assumption of (C3), TX is trivial. So in the following commutative diagram

$$\begin{array}{ccc} X\tilde{\Delta}L & \hookrightarrow & X \times |L| \\ & {\scriptstyle f_L} \searrow \quad \swarrow {\scriptstyle \pi_L} & \\ & |L| & \end{array}$$

we have that $T(X\tilde{\Delta}L)$ is stably equivalent to the pullback of TL. $\square$

Similarly, we have the smooth version,

Corollary 2.3. *Suppose that X satisfies (C1') and (C3) and that $|K|$ is a smooth n-manifold. Then the stable tangent vector bundle of $X\Delta K$ is the pullback of the stable tangent bundle of K, i.e.,*

$$T(X\Delta K) = (f_K)^* TK.$$

Remark 2.3. However, it's not guaranteed that $f_K : X\Delta K \to K$ can be covered by a map of unstable tangent bundles which is a fiberwise isomorphism, because $X\Delta K$ and K may have different Euler characteristics and hence the Euler classes of their tangent bundles may be different.

2.1.3 Relative Construction

In this section, suppose X satisfies (C1) or (C1').

We will try to construct a cobordism between $X\Delta K$ and K, for any K being a closed P.L. n-manifold.

Firstly, we introduce the relative version of the Williams functor on (K, J) [6, p. 358-359], where J is subcomplex of K. Intuitively, we just replace the simplex in $K' - J$ with (X, f), without changing the structure near J.

Let J be a subcomplex of K. Let $R(J, K)$ denote the standard derived neighborhood of J in K', $R^{\circ}(J, K)$ denote its relative interior, and $\partial R(J, K) = R(J, K) - R^{\circ}(J, K)$.

Let $\hat{K}$ denote the simplicial complex formed by deleting the interior of $R(J, K)$ from K' and attaching the cone on $\partial R(J, K)$, i.e.,

$$\hat{K} = (K' - R^{\circ}(J, K)) \cup \mathrm{Cone}(\partial R(J, K)).$$

Let c_0 be the cone point. The complex $K' - R^{\circ}(J, K)$ is a simplicial complex over σ^n. Moreover, under the map $d : K \to \{0, 1, \ldots, n\}$, no vertex of $\partial R(J, K)$ is mapped to the vertex of 0 in σ^n. Hence, the structure on $K' - R^{\circ}(J, K)$ as a complex over σ^n extends to a structure on $\hat{K}$ by sending c_0 to 0. Consider a point v_0 in $X\tilde{\Delta}\hat{K}$ which maps to c_0 in $\hat{K}$. By "lemma(1e1)", v_0 has a neighborhood in $X\tilde{\Delta}\hat{K}$ of the form $\mathrm{Cone}(\partial R(J, K))$. Remove the interior of this neighborhood and paste back $R(J, K)$. The result is denoted by $X\Delta(K, J)$ as the relative Williams construction on (K, J).

$$X\Delta(K, J) := (X\tilde{\Delta}\hat{K} - \mathrm{Dual}^{\circ}(v_0, X\tilde{\Delta}\hat{K})) \cup R(J, K).$$

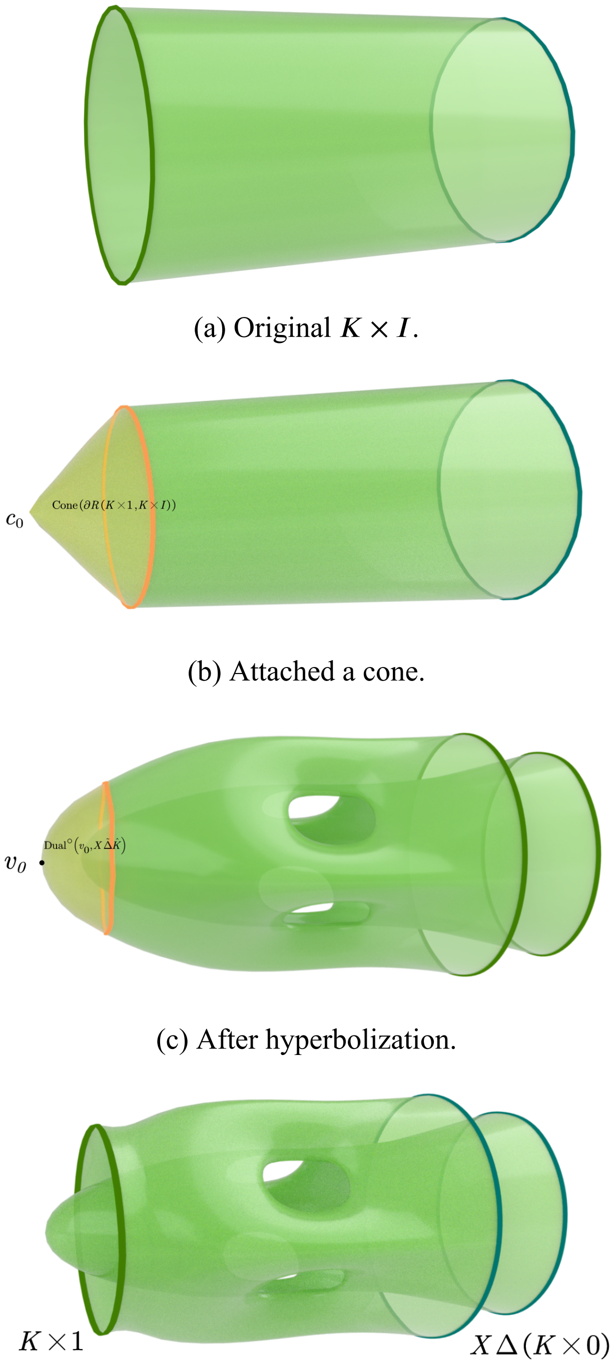
Remark 2.4. If K is a P.L. manifold, then $X\Delta(K, J)$ is also a P.L. manifold, similar to the argument in Corollary 2.1.

Theorem 2.3. *Suppose that K is a closed P.L. n-manifold, and (X, f) is exactly cover once onto σ^n (thus with degree 1), then $X\Delta(K \times I, K \times 1)$ is a cobordism between $X\Delta K$ and K.*

Proof. Although $K \times I$ is not a simplicial complex, but the relative construction still works because $(K \times I)'$ is a simplicial complex. Since $X\Delta(K \times I, K \times 1)$ is a P.L. $n+1$-manifold with boundary the components of the boundary are just $X\Delta(K \times 0)$ and $K \times 1$. □

Remark 2.5. The "exactly cover once" condition is essential, and it's why we always want the "degree one" condition (i.e. using the Gromov's construction of hyperbolized simplex) when establishing the "aspherical non-triangulable manifold" in Chapter 3. Otherwise, take (X, f) cover k times, then points in $X\Delta K$ corresponding to a point in $K \times 1$ would be k copies, and $X\Delta K$ could even be disconnected.

Example 2.3 (Hyperbolized cobordism but (X, f) not covering once)**.** Take K to be any simplicial complex of S^1, and (X, f) to be hyperbolized simplex as the second of Example 2.1. Then the boundary of $X\Delta(K \times I, K \times 1)$ is $S^1 \cong K \times 1$ one the one side, $S^1 \sqcup S^1 \cong X\Delta(K \times 0)$ on the other side. The process of relative construction is shown in Figure 6.



(a) Original $K \times I$.

(b) Attached a cone.

(c) After hyperbolization.

(d) Relatively hyperbolized cobordism.

Figure 6 The construction process of $X\Delta(K \times I, K \times 1)$.

Remark 2.6. Just for simplicity, genus showed on figure(c) and (d) is much less than the actual situation.

2.1.4 Asphericalization

Before doing hyperbolization, we have a weaker but very direct result to introduce, called asphericalization, to make a simplicial complex into a aspherical space.

Definition 2.2 (Aspherical)**.** Let X be a connected topological space, if $\pi_k(X) = 0, \forall k \geqslant 2$, then X is called an aspherical space.

We will show in next chapter that for non-positively curved space, asphericity is the corollary of generalized Cartan-Hadamard Theorem 2.5. And in the next theorem, the hyperbolized simplex constructed in section 2.3 is just example which satisfies (C4).

Theorem 2.4. *Suppose that* (L, π) *is a finite complex over* σ^n *and that* (X, f) *satisfies (C4), then* $X\tilde{\Delta}L$ *is aspherical. Moreover, if* J *is any subcomplex of* L *over* σ^n*, the inclusion induces a injection* $\pi_1(X\tilde{\Delta}J) \to \pi_1(X\tilde{\Delta}L)$.

This theorem can be proved in language of "graph of groups", which shares the flavor with van-Kampen Theorem. The proof can be found in [20, p. 156].

In simplicity, the theorem makes sense because that if two aspherical polyhedra intersect and there are injections from the π_1 of intersection to π_1 of the two polyhedra respectively, then the whole space is also aspherical.

2.2 Spaces of Non-positive Curvature

In this chapter, we introduce some basic concepts of metric geometry.

A geodesic segment in a metric space X is an isometric map from an interval to X. Therefore a triangle in X consists of three points together with three geodesic segments connecting them.

A metric space X is called geodesic if it is complete (the domain of every geodesic segment can be extended to $\mathbb{R}$) and if any two points in it can be connected by a geodesic segment. A subset Y of a geodesic space X is totally geodesic if locally every geodesic segment in X with endpoints in Y is actually contained in Y.

For each real number ε, let $M^2(\varepsilon)$ be the complete, simply connected Riemannian 2-manifold of sectional curvature ε. If T is a triangle in X, then a comparison triangle in $M^2(\varepsilon)$ is a triangle T' with the same edge lengths as T.

Remark 2.7. Comparison triangles always exist $\forall \varepsilon \leqslant 0$. If $\varepsilon > 0$, T has a comparison triangle provided T has perimeter $\leqslant 2\pi/\sqrt{\varepsilon}$.

Definition 2.3 (CAT(ε))**.** Suppose that T is a triangle in X with vertices x_0, x_1, x_2, and y is a point on the geodesic segment $[x_1, x_2]$. Let T' be a comparison triangle in $M^2(\varepsilon)$ with corresponding vertices x'_0, x'_1, x'_2, and y' be the point on $[x'_1, x'_2]$ corresponding to y, i.e. $d(y, x_i) = d'(y', x'_i)$, $i = 1, 2$. We say (T, y) satisfies CAT(ε) if $d(x_0, y) \leqslant d'(x'_0, y')$.

And we can make the following global definitions,

- The space X satisfies CAT(ε) if (T, y) satisfies CAT(ε) for any triangle T in X and point $y \in [x_1, x_2]$. (If $\varepsilon > 0$, then we only consider triangles of perimeter $\leqslant 2\pi/\sqrt{\varepsilon}$.)
- A geodesic space X has **curvature** $\leqslant \varepsilon$ if it satisfies CAT(ε) locally.

In the context of this article, a geodesic space X is called **non-positively curved**, if it satisfies CAT(0) locally.

Remark 2.8. Suppose X is a non-positively curved simply connected geodesic space, then X satisfies CAT(0) globally [10]. This implies that the distance function $d : X \times X \to \mathbb{R}$ is convex.

Theorem 2.5 (Generalized Cartan-Hadamard Theorem)**.** *A non-positively curved geodesic space is aspherical.*

Proof. Using the convexity of the distance function, we can prove that the universal cover of a non-positively curved geodesic space is contractible, thus itself is aspherical. □

The following lemma will guarantee that the hyperbolization procedure preserves non-positive curvature of each hyperbolized simplex. It's also used in the construction of hyperbolized simplex.

Theorem 2.6 (Gluing Lemma)**.** *Suppose that one of the following condition,*

1. *X is the disjoint union of two geodesic spaces X_1 and X_2 and that $Y_i \subset X_i$, $i = 1, 2$, is totally geodesic closed subspace, or*
2. *X is a geodesic space and Y_1, Y_2 are two disjoint totally geodesic closed subspaces.*

If $f : Y_1 \to Y_2$ is isometric, let $\widehat{X}$ be the space formed from X by identifying Y_1 with Y_2 via f. Then $\widehat{X}$ with natural metric is a geodesic space. Further, if the curvature of each component of X is $\leqslant \varepsilon$, with $\varepsilon \leqslant 0$, then the same is true for $\widehat{X}$.

Proof. Why $\widehat{X}$ is a geodesic space is a standard argument of metric geometry. We focus on the proof about the curvature.

What we should to prove is that $\widehat{X}$ is locally CAT(ε), given that X is locally CAT(ε). So we only concern about triangles on $\widehat{X}$ that are small enough. Suppose T is such a triangle with vertices x_0, x_1, x_2.

Denote $\widehat{Y}$ to be the image of Y_i. When x_0 falls into $\widehat{X} - \widehat{Y}$, T can be shrunk to $\widehat{X} - \widehat{Y}$ as a whole, so satisfies CAT(ε). Thus the crucial case is when $x_0 \in \widehat{Y}$, then $[x_0, x_1]$ and $[x_0, x_2]$ can be identified with geodesic segments in X. Suppose $[x_1, x_2]$ intersect $\widehat{Y}$ with segment $[y_1, y_2]$, and denote the triangles with vertex $\{x_0, y_1, y_2\}$, $\{x_0, x_1, y_1\}$, $\{x_0, x_2, y_2\}$ by T_0, T_1, T_2 respectively. Then denote T' and T_i', $i = 0, 1, 2$ to be comparison triangles in $M^2(\varepsilon)$ for T and T_i.

It is proved in [2, 4.10, p. 199] that angles at y_1' and y_2' are not convex, shown as in

Figure 7. Since $[x_1', x_2'] = [x_1', y_1'] + [y_1', y_2'] + [y_2', x_2']$, the three triangles T_i' can be seen as T' pulling $[x_1', x_2']$ closer to x_0'. In this way, for z on the segment opposite to x_0 on T_j, then $\tilde{z}$ on T_j' is not farther to x_0' than z' on T'. Thus we have $d(x_0, z) \leqslant d'(x_0', \tilde{z}) \leqslant d'(x_0', z')$. It is to say, T is a CAT(ε) triangle. □

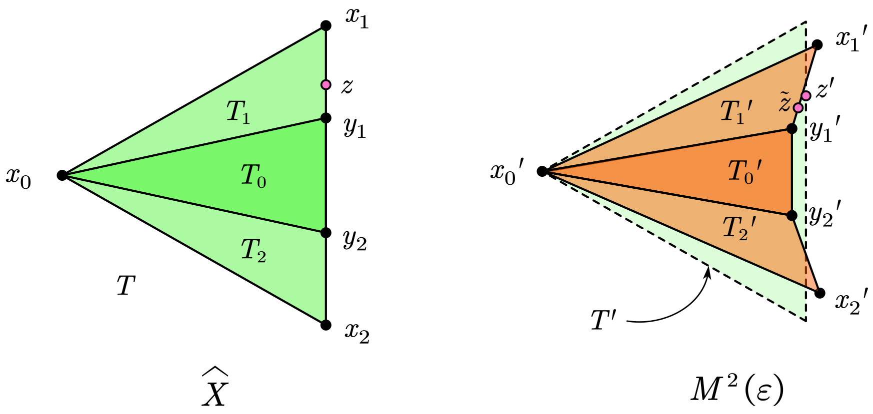


Figure 7 T' and T_i''s behavior on $M^2(\varepsilon)$.

Next we introduce some conceptions about the topology at infinity, which will be used to distinguish spaces by the infinity property of their universal covers.

For a point x in a metric space P, denote by $B_x(r)$ the open metric ball and $S_x(r)$ the metric sphere of radius r about x.

Suppose P is a CAT(0) geodesic space. Since the distance function is convex, any two points can be joined by a unique geodesic. Define a map $c_r : P - B_x(r) \to S_x(r)$, called geodesic contraction, by sending a point y to the point on the geodesic joining x and y of distance r from x. Besides, for $r \in (0, \infty)$, c_r is continuous deformation retraction.

Definition 2.4 (Visual sphere)**.** The visual sphere of P at x is the set of geodesic rays emanating from x, denoted by $S_x(\infty)$.

The definition says, $S_x(\infty)$ is just the inverse limit of $S_x(r)$'s, i.e. $S_x(\infty) = \varprojlim S_x(r)$.

Let P be a metric space, $C(P)$ be the space of continuous functions on P with topology of uniform convergence on compact sets.

Definition 2.5 (Ideal boundary)**.** Suppose that P is a CAT(0) geodesic space. Embed P into the space $C(P)$ by sending x to the function $d_x := d(x, \cdot)$. Let $L \subset C(P)$ be the linear subspace of constant functions. Then we can take closure $\bar{P}$ in the quotient space $C(P)/L$, and the ideal boundary of P is defined as $\bar{P} - P$.

Remark 2.9. The ideal boundary sometimes is also called the "visual sphere" or "sphere at infinity", although in our context the ideal boundary and the visual sphere are differently defined. However, the next theorem indicates that they could be the same in some circumstances.

Theorem 2.7 ([6, (2b.2)])**.** *Suppose that P is a* CAT(0) *geodesic space and that P is a*

Riemannian manifold on the complement of a set of codimension 2. Then for any $x \in P$, the natural map $\psi : S_x(\infty) \to \bar{P} - P$ is a homeomorphism.

2.2.1 Polyhedra of Piecewise Constant Curvature

Denote $M^n(\varepsilon)$ as the complete, simply connected Riemannian manifold of constant sectional curvature ε. Suppose K is an n-dimensional locally finite simplicial complex with vertex set V, and the function $\psi : V \to M^n$ maps vertices of any k-simplex α to $k+1$ points in $M^n(\varepsilon)$. Then we can identify each simplex in K with a geometric simplex in $M^n(\varepsilon)$ by ψ. Let P denote the geometric realization of K, then P has intrinsic metric d induced by ψ, and (P, d) is called a polyhedron of piecewise constant curvature ε.

P is called piecewise spherical, piecewise flat or piecewise hyperbolic as ε equals $+1, 0, -1$ respectively.

Example 2.4 (Natural piecewise spherical structure of links)**.** Suppose that α is geometric n-simplex in $M^n(\varepsilon)$ and v is a vertex of α. Since the geodesic rays emanating from v can intersect α at at most one point, $\text{Link}(v, \alpha)$ can be mapped homeomorphically to a geodesic sphere centered at v. Thus $\text{Link}(v, \alpha)$ has natural spherical structure pulled back from geodesic sphere, and link of v in P has a natural piecewise spherical structure.

Remark 2.10. Similarly, for k-simplex β in P, $\text{Link}(\beta, P)$ has natural piecewise spherical structure pulled back from a $n-k-1$ dimensional sphere.

If β is k-face of n-simplex α, and $x \in \beta$, then $\text{Link}(x, \alpha)$ is the $(k-1)$-fold suspension of $\text{Link}(\beta, \alpha)$.

Definition 2.6 (Large spherical polyhedra)**.** A piecewise spherical polyhedron L is **large** if any two points x and y in L with $d(x, y) < \pi$ can be joined by a unique geodesic segment in L.

For example, if L is homeomorphic to a circle, then L is large if and only if its circumference is $\geqslant 2\pi$.

Except for our combinatorial definition of link, we can also define the geometric link of $\beta \in K$ to be the union of the end points of geodesic segments of small length emanating perpendicularly to β from some point $x \in \mathring{\beta}$. These two definitions are actually equivalent in our setting. So if we are allowed to confuse the definition, we can define $\text{Link}(x, P)$ for $x \in P = |K|$ in the geometric sense.

Lemma 2.4. *Let $P = |K|$ be a piecewise constant curvature polyhedron. The following statements are equivalent:*

1. $\text{Link}(x, P)$ *is large for all $x \in P$.*
2. $\text{Link}(\beta, K)$ *is large for each simplex β of K.*

*If either condition of the lemma holds, we say that "P **has large links**".*

Proof. The equivalence comes from the fact that a piecewise spherical polyhedron is large if and only if its k-fold suspension is large. □

We introduce the following conclusion of "large link" in [6] without proof.

Theorem 2.8. *Suppose that P is a polyhedron of piecewise constant curvature $\varepsilon \leqslant 0$. Then the curvature of P is $\leqslant \varepsilon$ if and only if P has large links.*

Lemma 2.5. *Suppose that L is a large piecewise spherical polyhrdron which is a P.L. n-manifold. Then for any $v \in L$ and $r \in (0, \pi)$, $\bar{B}_v(r)$ is homeomorphic to the standard closed n-ball. Consequently $B_v(\pi)$ is homeomorphic to an open n-ball.*

Theorem 2.9. *Suppose that Q is a P.L. n-manifold and is a simply connected, piecewise flat polyhedron with large links, then*

1. *For each $x \in Q$ and $r \in (0, \infty)$, $\bar{B}_x(r)$ is homeomorphic to the standard n-ball.*
2. *Q is homeomorphic to $\mathbb{R}^n$.*
3. *The visual sphere $S_x(\infty)$ is homeomorphic to S^{n-1}.*

For the non-P.L. case, we have:

Theorem 2.10. *Suppose that Q is a polyhedral homology n-manifold and that Q is simply connected, piecewise flat polyhedron with large links. Then for each $x \in Q$ and $r \in (0, \infty)$, $\bar{B}_x(r)$ is contractible homology n-manifold with boundary $S_x(r)$, then we have the following properties,*

1. *$S_x(r)$ is a generalized homology $(n-1)$-sphere.*
2. *If $s > r$, then geodesic contraction $c_r : S_x(s) \to S_x(r)$ is a map of degree one. Hence the induced map on fundamental groups is surjective.*
3. *The fundamental group at infinity of Q is the inverse limit*

$$\pi_1^{\infty} = \varprojlim \pi_1(S_x(r)).$$

Remark 2.11. The ideal boundary of the universal cover is a finer invariant than its fundamental group at infinity.

Corollary 2.4. *Suppose Q is as above. If there exists a point $x \in Q$ and a real number r such that $S_x(r)$ is not simply connected, then Q is not simply connected at ∞.*

Theorem 2.11. *Suppose Q as above, and the P.L. singular set of Q is discrete. Let $s_1, \ldots, s_k$ denote the P.L. singular points in $B_x(r)$, where the r is chosen avoiding $S_x(r)$ contain singular points. Then*

$$S_x(r) = \mathrm{Link}(s_1, Q)\# \ldots \#\mathrm{Link}(s_k, Q).$$

So once Q has any P.L. singular point, it can't be simply connected at ∞, because of Corollary 2.4.

2.3 Hyperbolization

In this section, we formally introduce the hyperbolization process.

Simply by gluing lemma 2.6, we have a direct corollary, can be seen as an initial version of hyperbolization.

Theorem 2.12. *Suppose that (L, π) is a finite simplicial complex over σ^n, and (X, f) is a space over σ^n satisfying (C5). Then $X\tilde{\Delta}L$ is a geodesic space of curvature $\leqslant 0$. Moreover, if P is any connected subcomplex of L, then $X\tilde{\Delta}P$ is a totally geodesic subspace of $X\tilde{\Delta}L$.*

For practical use, we need (X, f) to be tame as manifold. So we just add in (C1) to define **hyperbolized simplex**.

Definition 2.7. Suppose that (X, f) is a space over σ^n. Then (X, f) is called a hyperbolized n-simplex if is satisfies conditions (C1) and (C5). It is strictly hyperbolized if its curvature is strictly negative.

Combining Theorem 2.12, Theorem 2.2, Theorem 2.3, Corollary 2.1, and Corollary2.3 for (X, f) satisfies all conditions (C0) to (C5), we have the main theorem of this Chapter.

Theorem 2.13 (Davis & Januszkiewicz)**.** *Suppose that (X^n, f) is a hyperbolized n-simplex which is degree one and tangentially trivial. For any n-dimensional simplicial complex K, let $a(K) := X\Delta K$, then*

1. *$a(K)$ is non-positively curved, thus aspherical space,*
2. *$(f_K)_* : H_*(a(K)) \to H_*(K)$ is surjective,*

If $|K|$ is an n-manifold, so is $a(K)$, and

3. *$a(K)$ is cobordant to $|K|$.*
4. *The stable tangent bundle of $a(K)$ is the pullback of that of $|K|$ via f_K.*

Now we introduce certain ways to construct hyperbolized simplices.

2.3.1 Cartesian Product with an Interval

It's a direct construction, and constructed by induction on dimension n.

When $n = 1$, just choose X^1 to be the one-simplex. Suppose that (X^n, f) is a tangentially trivial, hyperbolized n-simplex. Let $Y^n = X^n\Delta(\partial\sigma^{n+1})$. Let $X^{n+1} = Y^n \times [0, 1]$.

Then we need to define the map $f : X^{n+1} \to \sigma^{n+1}$. Since the boundary of X^{n+1} is two copies of Y^n, there is natural projection $\partial X^{n+1} = Y^n \times \{0, 1\} \to \partial\sigma^{n+1}$. So the restriction of f to ∂X^{n+1} can be defined to be the projection, and then extend to $X^{n+1} \to \sigma^{n+1}$. Since X^n is non-postively curved, so is Y^n because of the gluing lemma 2.6. So X^{n+1} is also non-positively curved as product space of two non-positively curved spaces.

Now (X^{n+1}, f) has been constructed as we want.

Remark 2.12. (X^2, f) here is just the second of Example 2.1, thus it's degree 0. Actually, since (X^n, f) in all dimension is product of interval, it's also degree 0 in all dimension.

To have degree 1 hyperbolized simplex which is for Theorem 2.13, we need a more precise construction, shown following as the Gromov's construction.

2.3.2 Gromov's Construction

By Gromov's construction, we could construct (X, f) that is a piecewise flat, tangentially trivial, **degree one** and hyperbolized n-simplex in all dimension n, i.e. satisfying conditions (C1) (C2') (C3) (C5).

Suppose that r is a reflection on a space Y, A is a half-space, and $B = A \cap r(A)$ is the fixed point set.

Let $\Omega(Y, A, r)$ be the space formed from $Y \times [-1, 1]$ by gluing $r(A) \times -1$ to $r(A) \times 1$, then the image of $A \times \{\pm 1\}$ in $\Omega(Y, A, r)$ denoted by $\partial\Omega$ is naturally identified with Y.

Proposition 2.2. *Suppose that r is a reflection on a space Y and A is a half-space for r. Let $\Omega = \Omega(Y, A, r)$.*

1. *Suppose that Y is an n-dimensional manifold and that r is locally linear. Then Ω is an $(n + 1)$-manifold with boundary $\partial\Omega = Y$.*
2. *Suppose Y is aspherical, then Ω is aspherical. Moreover, $\pi_1(Y) \to \pi_1(\Omega)$ is an injection.*
3. *Suppose that Y is a geodesic space and r is an isometric reflection. Then the induced metric on Ω makes Ω into a geodesic space and $\partial\Omega$ is a totally geodesic subspace. Suppose further that the curvature of Y is non-positive, then the curvature of Ω is non-positive.*

Proof. 1. needs only to check definition. 2. is proved similarly as Theorem 2.4. 3. is proved by the Gluing lemma 2.6. □

Remark 2.13. $\Omega(Y, A, r)$ is also has trivial stable tangent bundle if A has. The proof can be found in [6, (4c.2)]

The construction. It's also constructed by induction. Assume by induction that (X^n, f) is a piecewise flat, tangentially trivial, degree one, hyperbolized n-simplex. Let $Y^n = X^n\Delta(\partial\sigma^{n+1})$ (Warning: there is not $\tilde{\Delta}$.). The automorphism group of σ^{n+1} is Σ_{n+1}, the symmetric group of degree $n + 2$. It acts on the derived complex $(\partial\sigma^{n+1})'$ through automorphisms over σ^n and the natural map $\pi : (\partial\sigma^{n+1})' \to \sigma^n$ can be identified with the orbit map $S^n \to S^n/\Sigma_{n+2} \cong \sigma^n$. By functoriality of the Williams construction, Σ_{n+2} acts on Y^n and the natural map $Y^n \to (\partial\sigma^{n+1})$ is Σ_{n+2}-equivariant. Suppose r is a transposition in Σ_{n+2}, then r is a reflection acting on Y^n. Suppose A^n is the half space for r on Y^n, then let $X^{n+1} = \Omega(Y^n, A^n, r)$.

Extend the natural map $\partial X^{n+1} = Y^n \to \partial\sigma^{n+1}$ to $f : X^{n+1} \to \sigma^{n+1}$ by choosing a collared neighborhood of ∂X^{n+1} in X^{n+1}. By the fact that σ^{n+1} is the cone on $\partial\sigma^{n+1}$, the extension works.

Then (X^n, f) is constructed inductively with the properties we need.

Remark 2.14. (X^2, f) of Gromov's construction is just the first of Example 2.1.

2.3.3 Stronger Results

The above hyperbolized simplex needs only to be curvature $\leqslant 0$. However, it can be improved to be curvature $\leqslant -1$ by certain construction, that is to say, we can apply "strict hyperbolization". This result is given by Charney and Davis [4].

A more recent result based on strict hyperbolization by Ontaneda [18] in 2010s shows that if $|K|$ is smooth manifold, then the product after strict hyperbolization can be realized to be a Riemannian manifold, by forcing the original metric to be smooth inductively, using specific warp product technique of Riemannian Geometry. This process is called Riemannian hyperbolization, which can be seen as a smooth version of Corollary 2.1 (actually a little weaker, because Riemannian hyperbolization requires (X, f) to be strict hyperbolized, rather than any given smooth manifold with corners). Further, the resulting manifolds can fall into a ε-pinch.

Theorem 2.14 (Ontaneda)**.** *Let M^n be a closed smooth manifold and $\varepsilon > 0$. Then there is a closed Riemannian manifold N^n and a smooth map $f : N^n \to M^n$ such that N^n is homeomorphic to the strict hyperbolization of M^n, and N^n has sectional curvatures in the interval $[-1-\varepsilon, -1]$. Moreover, $f_* : H_*(N) \to H_*(M)$ is surjective and f^* pulls back the rational Pontryagin classes.*

Chapter 3 Non-Triangulable Manifolds

As we have seen, the polyhedra can always be hyperbolized. The simplicial complex structure on a polyhedron is called a **triangulation**. Thus if we want to make some space into non-positively curved, to find a triangulation is a shortcut.

A natural question is that "Does the triangulable spaces exist widely?". Obviously, triangulation can't exist on arbitrary topological spaces, but how about some more well-behaved spaces, like manifolds? Then it encounters a famous problem in Geometric Topology, the Triangulation Conjecture.

3.1 Triangulation Conjecture

From now on, manifolds are assumed to be closed, unless otherwise stated.

Definition 3.1 (Triangulation of manifold)**.** Assume M^n is n dimensional topological manifold, we say it is **triangulable** if and only if there exists a homeomorphism $\varphi : |K| \to M^n$, where K is a simplicial complex.

Manifolds are locally Euclidean spaces, i.e. they are locally triangulable. It's reasonable that this good local behavior can be patched into a global property, that is to say, manifold admit triangulation.

This is just the conjecture proposed by Kneser in 1926.

Conjecture 3.1 (Triangulation Conjecture)**.** *Any topological manifold is also a polyhedron, i.e. admits triangulation.*

Early in the 1930s, Cairns [3] and Whitehead [22] proved that all smooth manifolds admit triangulation (even combinatorial triangulation, which will be defined following).

However, the general answer is NO. The complete conclusion was finally drawn by Manolescu in 2013 [15], achieved through modern methods in gauge theory.

During the period when triangulation conjecture was widely open, people studied a stronger version of triangulation simultaneously.

Definition 3.2 (Combinatorial triangulation)**.** We say a manifold M^n is combinatorial triangulable if and only if there exists the triangulation $\varphi : |K| \to M^n$ and $\forall \sigma^q \in K$, the link of σ^q is P.L. homeomorphic to the standard sphere.

In fact, the existence of combinatorial triangulation on manifold is equivalent to the P.L. structure of manifold (P.L. to manifold is defined like smooth to manifold. Just to replace the word "smooth" in definitions into "piecewise linear".).

The combinatorial triangulation is strictly stronger than triangulation, example as the double suspension of Poincaré homology sphere. However, the two kinds of triangulation are closely connected.

In the 1960s, Kirby and Siebenmann [11][12] proved the existence of the obstruction to the P.L. structure on topological manifold for dimension $n \geqslant 5$, called **Kirby-Siebenmann invariant** $\Delta(M) \in H^4(M;\mathbb{Z}/2)$, satisfying the following theorem.

Theorem 3.1. *Suppose M^n is a topological manifold of dimension $n \geqslant 5$. Then M admits P.L. structure if and only if $\Delta(M) = 0$. Further, if $\Delta(M) = 0$, the non-equivalent P.L. structures on M are parametrized by $H^3(M;\mathbb{Z}/2)$.*

The obstruction of triangulation comes from the Kirby-Siebenmann invariant as follows.

The short exact sequence

$$0 \to \ker \mu \xrightarrow{\iota} \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2 \to 0$$

where

1. Θ_3 is the homology cobordism group, containing equivalence classes of integral homology 3-spheres. The equivalence relation is defined as follows: regard two integral homology Y_1, Y_2 equivalent if there is a smooth, oriented, compact 4-manifold M with boundary $\partial M = -Y_1 \sqcup Y_2$ and $H_*(M) = H_*(S^3 \times [0,1])$. The group structure is given by connected sum $[Y_1] + [Y_2] := [Y_1 \# Y_2]$ as addition, orientation reversing $-[Y_1] := [-Y_1]$ as inverse, and S^3 as the identity element.
2. $\mu : \Theta_3 \to \mathbb{Z}/2$ is the Rokhlin homomorphism, defined as follows. For a homology sphere Y, it bound a compact spin 4-manifold W, and the signature $\sigma(W)$ (the number of positive eigenvalues minus the number of negative eigenvalues) is divisible by 8. Then $\mu(Y) := \sigma(W)/8 \mod 2$. By Rokhlin's theorem 3.4, μ is independent of the choice of W, and actually a homeomorphism. Since D^4 is countractible, $\mu(S^3) = 0$. And Poincaré homology sphere is the boundary of E_8 plumbing (E_8 matrix is 8 by 8 and all eigenvalues are positive), so $\mu(P^3) = 1$, which indicates that Θ_3 is non-trivial.

induces the long exact sequence of cohomology groups with coefficients

$$\cdots \xrightarrow{\delta} H^4(M;\ker\mu) \xrightarrow{\iota} H^4(M;\Theta_3) \xrightarrow{\mu} H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\ker\mu) \xrightarrow{\iota} \cdots,$$

where δ is the Bockstein homomorphism.

Galewski & Stern [9] and Matumoto [16] in the 1970s proved that $\delta(\Delta(M)) \in H^5(M;\ker\mu)$ is just the obstruction of triangulation for $n \geqslant 5$.

Theorem 3.2. *Suppose M^n is a topological manifold of dimension $n \geqslant 5$. Then M admits triangulation if and only if $\delta(\Delta(M)) = 0 \in H^5(M;\ker\mu)$. If $\delta(\Delta(M)) = 0$, the non-equivalent triangulations are parametrized by $H^4(M;\ker\mu)$.*

In language of categories, the four types of manifolds, topological, triangulable, P.L., and smooth, obey the strict strong-weak relationship shown in Figure 8.

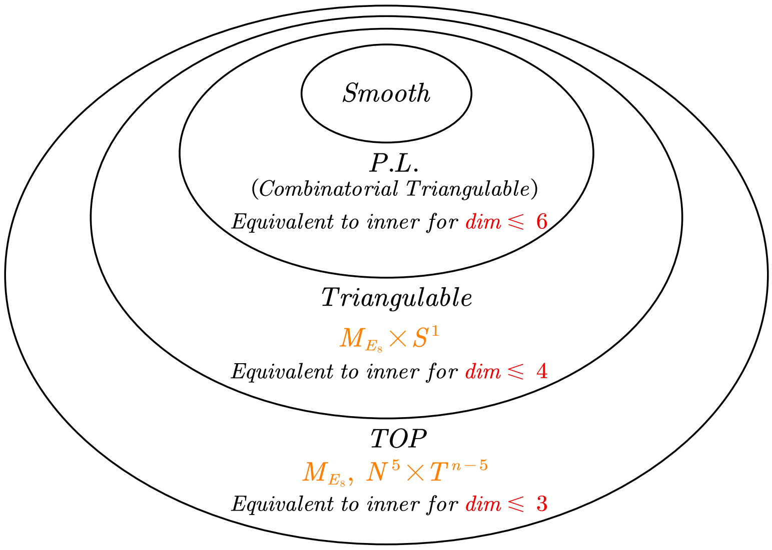


Figure 8 Relationship of four categories of manifolds.

In dimension $n \leqslant 3$, they are all equivalent (proved by Moise [17]), i.e. all topological manifolds can be triangulated and combinatorial triangulated and admit smooth structure.

Dimension 4 is a gap between $\leqslant 3$, and $\geqslant 5$ where the obstruction of triangulation works. But interestingly, the first non-triangulable example, the Freedman's E_8 manifold M_{E_8}, was given in dimension 4. It's constructed by Freedman in 1982 [7], and proved to be non-triangulable by Casson in 1990 [1].

Until 2013, based on the results of Galewski & Stern, Manolescu proved that in each dimension $\geqslant 5$, there also exists non-triangulable manifolds[14][21].

So we conclude the theorem,

Theorem 3.3. *There exists non-triangulable closed manifold in each dimension* $n \geqslant 4$.

Next we will give examples in dimension $n = 4$ and $\geqslant 5$. As the cornerstone in all dimensions, we start with $n = 4$.

3.2 Freedman's E_8 4-Manifold

According to the Dynkin diagram E_8, we can identify the particular subsets of unit disk bundles of S^2, $\mathbb{D}TS^2$ by the following "plumbing" rule (sketched in [19, p. 86]).

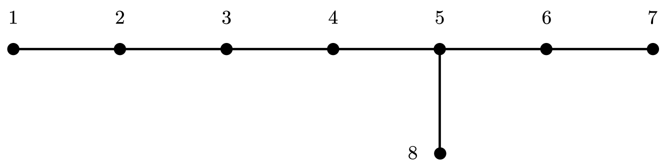


Figure 9 The E_8 Dynkin diagram.

Each node of the E_8 diagram represents a $\mathbb{D}TS^2$, where S^2 has self intersection number -2. For two adjacent node, take disks D', D'' on each S^2, then identify the sub bundle $D' \times \mathbb{D}^2$ with $\mathbb{D}^2 \times D''$ factor by factor. In this way, two S^2's in each adjacent $\mathbb{D}TS^2$ after identification, have exactly intersection number 1.

After the construction, we get a smooth, simply connected 4-manifold $Q(E_8)$ with intersection form to be E_8 matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

and with boundary P^3, the Poincaré homology sphere.

By the work of Freedman (proved by using Casson handle), we know that P^3 bounds a contractible 4-manifold F^4 [7]. Then attach F^4 to $Q(E_8)$ along P^3, we get the Freedman's E_8 manifold, a closed 4-manifold denoted by M_{E_8}.

Rather than directly attaching a cone on P^3 to get a homology manifold with cone point as P.L. singularity, this process of attaching contractible manifold is called "topological resolution of the singularity".

Theorem 3.4 (Rokhlin)**.** *Assume M^4 is a orientable closed P.L. manifold with $w_2 = 0$, then $\sigma(M)$ is divisible by* 16.

By Rokhlin's theorem, M_{E_8} admit no P.L. structure. If not, $\sigma(M_{E_8}) = 8$ is a contradiction.

Further, by Casson's invariant [1], it also can be proved that M_{E_8} admits no triangulation.

In fact, triangulation structure is equivalent to P.L. structure (combinatorial triangulation) in dimension 4, as shown in Figure 8. This fact can be proved by Casson's invariant, and also can be proved by the Poincaré Conjecture: for any 4-dimensional triangulation, the link of a simplex of dimension > 0 is a homology sphere of dimension $\leqslant 2$, hence a standard sphere. And the link of a vertex is a simply connected (because the punctured star neighborhood is $\mathrm{Link}(p) \times (0,1) \cong S^3 \times (0,1)$, thus $\pi_1(\mathrm{Link}(p)) = \pi_1(S^3) = 0$) homology 3-sphere, hence a homotopy sphere, and by the Poincaré Conjecture, a standard 3-sphere.

Remark 3.1. $M_{E_8} \times T^{n-4}$, $n \geqslant 5$ still admits no P.L. structure because their $\Delta \neq 0$. However, they admits triangulation. Let $K(E_8) = \mathrm{Cone}(P) \cup Q(E_8)$ be the homology manifold by directly attaching a cone on the boundary of Q_{E_8}. So $K(E_8)$ admits triangulation. And in fact, $M_{E_8} \times S^1 \cong K(E_8) \times S^1$, so $M_{E_8} \times S^1$ can be triangulated by the natural triangulation of $K(E_8) \times S^1$, and so do the other $M_{E_8} \times T^{n-4}$'s.

3.3 The Universal 5-Manifold

For $n = 5$, Galewski and Stern constructed a universal 5-manifold N^5 [8], where the term "universal" means that if N^5 is triangulable, then all manifolds in dimension $\geqslant 5$ are triangulable. Thus if there exist examples non-triangulable in dimension 5, N^5 must be one, so as $N^5 \times T^{n-5}$ in all dimensions $n \geqslant 5$.

Finally, Manolescu proved that there exists non-triangulable closed manifold in dimension $n \geqslant 5$ in 2013, which drew conclusion to the triangulation conjecture.

So in this section, we mainly introduce the Galewski and Stern's work.

In dimension $n \geqslant 5$, we have a direct corollary from Theorem 3.2.

Theorem 3.5. *If the short exact sequence*

$$0 \to \ker(\mu) \xrightarrow{\iota} \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2 \to 0 \tag{3-1}$$

splits, then all closed topological manifolds with dimension $n \geqslant 5$ admit triangulation.

Proof. Given that short exact sequence splits, there exists injection $k : \mathbb{Z}/2 \to \Theta_3$ such that $\mu \circ k = \mathrm{id}_{\mathbb{Z}/2}$. Then $\Theta_3 = \ker \mu \oplus k(\mathbb{Z}/2)$, and $\delta(\Delta(M)) = \iota^{-1} \circ \partial^*(k(\Delta(M)))$, where the coefficient of $\partial^*(k(\Delta(M)))$ falls into $k(\mathbb{Z}/2)$ part of Θ_3, thus $\delta(\Delta(M)) = 0$, for any M being a topological manifold with dimension $n \geqslant 5$. By Theorem 3.2, we know that M admits triangulation. $\square$

Actually, the converse direction of this theorem is also correct. Galewski and Stern constructed a "universal 5-manifold" N^5, satisfying that if N^5 is triangulable, the short exact sequence 3-1 is split. If so, by applying Theorem 3.5, all manifolds in dimension $\geqslant 5$ would be triangulable, where is the term "universal" comes from.

To illustrate how universal 5-manifold works, we need some algebraic preparations.

The short exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{r} \mathbb{Z}/2 \to 0$$

induces a long exact sequence

$$\cdots \xrightarrow{\beta} H^i(M; \mathbb{Z}/2) \xrightarrow{\times 2} H^i(M; \mathbb{Z}/4) \xrightarrow{r} H^i(M; \mathbb{Z}/2) \xrightarrow{\beta} H^{i+1}(M; \mathbb{Z}/2) \xrightarrow{\times 2} \cdots$$

where the Bockstein homomorphism β is just the first Steenrod square $\mathrm{Sq}^1 = \beta : H^k(M; \mathbb{Z}/2) \to H^{k+1}(M; \mathbb{Z}/2)$.

The "universal 5-manifold" N^5 by Galewski & Stern satisfying $\mathrm{Sq}^1(\Delta(N)) \neq 0$. With the help of N^5, we have the converse:

Theorem 3.6. *If the "Universal 5-manifold" N^5 can be triangulated, the short exact sequence 3-1 splits.*

Proof. It's proved by contradiction. Assume the short exact sequence 3-1 does not split. Then Θ_3 contains no element with $\mu = 1$ and of order 2, i.e. $\forall [Y] \in \Theta_3$ satisfying $\mu(Y) = 1$, we have $2[Y] \neq 0$. Let Θ' be the group generated by all 3-dimensional links in a given triangulation of N, as a subgroup of Θ_3. Let $i : \Theta' \hookrightarrow \Theta_3$ be the inclusion map.

Since the triangulation on N contains finitely many 3-dimensional links, Θ' can be written as a direct sum of finitely many cyclic groups, $\Theta' = \langle h_1 \rangle \oplus \cdots \oplus \langle h_k \rangle$ where each term is either a free cyclic group or a finite cyclic group of prime power order.

Next we define a map $\gamma : \Theta' \to \mathbb{Z}/4$. We only need to define it on $\{h_i\}_{i=1}^k$: 1' If $\mu(h_i) = 0$, let $\gamma(h_i) = 0$. 2' If $\mu(h_i) = 1$ and $\langle h_i \rangle \cong \mathbb{Z}$, let $\gamma(h_i) = \langle h_i \rangle \mod 4$; 3' If $\mu(h_i) = 1$ and the order of h_i is p^m, $p = 2$ because of $\mu \in \mathbb{Z}/2$ is order 2, further since Θ_3 contains no element of order 2, we have $m \geqslant 2$. Thus we can also let $\gamma(h_i) = \langle h_i \rangle \mod 4$. By this definition, we have $\mu \circ i = r \circ \gamma$.

For triangulable manifold, the obstruction of P.L. structure can be expressed in Θ_3 coefficient by adding up all codimensional simplices. And take the the Poincaré dual, we have $c(N) = \text{dual}\left(\sum_\sigma [lk(\Sigma)]\dot{\sigma}\right) \in H^4(N;\Theta_3)$, satisfying $\mu(c(N)) = \Delta(N)$. Besides, $\exists c'(N) \in H^4(N;\Theta')$ such that $i(c'(N)) = c(N)$. Thus

$$\mathrm{Sq}^1(\mu(c(N))) = \mathrm{Sq}^1(\mu(i(c'(N)))) = \mathrm{Sq}^1(r(\gamma(c'(N)))) = 0,$$

since in the long exact sequence $\mathrm{Sq}^1 \circ r = 0$. So we have $\mathrm{Sq}^1(\Delta(N)) = \mathrm{Sq}^1(\mu(c(N))) = 0$, which contradicts that $\mathrm{Sq}^1(\Delta(N)) \neq 0$. □

Since the 5-dimensional manifold $N^5 \times T^{n-5}$ also satisfies $\mathrm{Sq}^1(\Delta) \neq 0$, we have

Theorem 3.7. *If the short exact sequence does not split, on each dimension $n \geqslant 5$, there exist non-triangulable manifolds, and $N^5 \times T^{n-5}$ is such an example.*

The construction of N^5. We start with the E_8 plumbing $Q(E_8)$. Attach a cone to the boundary $P^3 = \partial Q(E_8)$, we get a homology manifold $K(E_8) = \mathrm{Cone}(P^3) \cup Q(E_8)$. Then we attach a orientation-reversing 1-handle $D^3 \times I$ to $P^3 \times 0$ and $P^3 \times 1$ in $K(E_8) \times I$, working as "boundary connected sum" of $Q(E_8) \times 0$ and $Q(E_8) \times 1$, whose boundary now becomes $P^3 \# P^3$. The boundary of $\mathrm{Cone}(P^3) \cup (D^3 \times I) \cup \mathrm{Cone}(P^3)$ is also $P^3 \# P^3$, so we can fill in the boundary with $\mathrm{Cone}(P^3 \# P^3)$ to obtain a homology 4-sphere T. Next fill in T with $\mathrm{Cone}(T)$, by *some double suspension like argument*, the interior of $\mathrm{Cone}(T)$ is 5-manifold. Thus

$$R^5 := (X^4 \times I) \cup \mathrm{Cone}(T)$$

is a polyhedral 5 manifold with boundary $\mathrm{Cone}(P^3 \# P^3) \cup Q(E_8) \#_b Q(E_8)$, and ∂P contains a single non-manifold point, the cone point of Cone($P^3 \# P^3$).

The Kirby-Siebenmann invariant $\Delta(R^5) \neq 0$ is the image of $[K(E_8)]$ in $H^4(R^5; \mathbb{Z}/2)$, thus R does not admit a P.L. structure. Since $\Delta(\partial R)$ is 0, $\Delta(R)$ is also the image of $\Delta(R, \partial R) \in H^4(R, \partial R; \mathbb{Z}/2)$.

Next we try to make R to be a closed manifold and to find the newly constructed manifold is non-triangulable. However, just in this step, the original construction of Galewski and Stern is hard to work on hyperbolizing. We will explain in next chapter.

Attach an external collar $\partial R \times [0, 1]$ to R, obtain a P.L. manifold V^4 embedded in $\partial R \times (0, 1)$ which separates $\partial R \times (0, 1)$ into two parts, and then define U to be the part of external collar between $\partial R \times 0$ and V^4. There V^4 bounds a P.L. 5-manifold W, thus we glue in W to get

$$N^5 := R \cup U \cup W$$

as we desired, because by Wu's formula, we have

$$\mathrm{Sq}^1(\Delta(N)) = \mathrm{Sq}^1(\Delta(R, \partial R)) = w_1 \smile \Delta(R, \partial R) \neq 0.$$

The non-degeneracy of the first Stiefel-Whitney class w_1 comes from the orientation-reversing 1-handle.

Chapter 4 Aspherical Non-triangulable Manifolds

We mostly desire to construct a CAT(0) non-triangulable manifold. However, it's hard to track much geometry in the process of topological construction. Thus we trade off to ignore the geometry, and try to obtain properties invariant under homotopy equivalent, for example, asphericity.

Combining the techniques in Chapter 2 and Chapter 3, we can construct aspherical non-triangulable manifold as follows.

4.1 In Dimension 4

Once again, we start with dimension 4. This example was given by Davis and Januszkiewicz [6, (5a)].

Take the homology 4-manifold $K(E_8) = \text{Cone}(P^3) \cup Q(E_8)$ (the cone point is the only non-manifold point, whose link is P^3, not S^3).

Let (X^4, f) be the hyperbolization of the 4-simplex by Gromov's construction, and

$$G^4 := X^4 \Delta K(E_8),$$

so G^4 is non-positively curved (hence aspherical) homology 4-manifold. And because (X, f) is degree 1, the original cone point has only one copy in G^4, also as the only non-manifold point of G^4.

Lemma 4.1. *G^4 is oriented, with $w_2 = 0$ and signature of the intersection form $\sigma(G^4) = 8$.*

Proof. Because G^4 and $K(E_8)$ "has the same link" by Theorem 2.2, G^4 is also a oriented homology 4-manifold with $w_2 = 0$, the same as $K(E_8)$.

And since G^4 is cobordant with $K(E_8)$ by Theorem 2.3, $\sigma(G^4) = 8$, the same as $K(E_8)$, because of Thom's theorem for $4k$ dimensional boundary of $4k + 1$ dimensional cobordism. □

Also because of G^4 has links isomorphic to $K(E_8)$'s, after removing a regular neighborhood of the cone point of G^4, the boundary is also P^3. Thus we can replace the neighborhood with the contractible 4-manifold F^4 again, then we get a closed 4-manifold N^4, which is homotopy equivalent to G^4.

Theorem 4.1. *The closed 4-manifold N^4 constructed above has the following properties.*

1. *N^4 is aspherical,*
2. *N^4 is not homotopy equivalent to a P.L. 4-manifold,*
3. *N^4 is not homeomorphic to a simplicial complex,*

4. *The universal cover $\tilde{N}^4$ is not simply connected at infinity. (Hence $\tilde{N}^4$ is not homeomorphic to $\mathbb{R}^4$.)*

Proof.

1. Since N^4 is homotopy equivalent to G^4, N^4 shares the same homotopy group with G^4, and is also aspherical.
2. N^4 is orientable, with $w_2 = 0$, and $\sigma(N^4) = 8$, so if N^4 is homotopy equivalent to some P.L. 4-manifold, it will inherit these numbers, which contradicts to Rokhlin's theorem requiring $16 \mid \sigma(N^4)$.
3. It is proved by Casson's invariant also because N^4 is orientable, and with $w_2 = 0$, $\sigma(N^4) = 8$.
4. It suffices to prove that G^4 has this property. Because G^4 contains the cone point of P^3, the P.L. singular point and its corresponding points in its universal cover $\tilde{G}$ are also P.L. singular. By Corollary 2.4 and Theorem 2.11, $\tilde{G}$ is not simply connected at infinity.

□

Remark 4.1. If we apply strict hyperbolization [4] on $K(E_8)$, we can further smooth the resulting G^4 by Ontaneda's smoothing process [18] to get a homology manifold with a single non-manifold point. Then we can cut off a regular neighborhood of this point with any given small size, to get a negatively curved (ϵ-pinched to -1) Riemannian manifold with boundary P^3. This item could be of use for future studies.

4.2 In Dimension $n \geqslant 6$

This construction is given by Davis, Fowler, and Lafont [5] in 2013, just after Manulescu's result of non-triangulable $n \geqslant 5$-manifolds.

Theorem 4.2 (Davis, Fowler, Lafont)**.** *There exists a closed aspherical non-triangulable manifold in each dimension $n \geqslant 6$.*

Modification of Galewski & Stern's construction. In dimension $n \geqslant 6$, we have more room to maintain P.L. structure for hyperbolization.

We take dimension 6 as an example. Let $R' := R^5 \times S^1$, then

$$\Delta(R') \in H^4(R; \mathbb{Z}/2) \otimes H^0(S^1; \mathbb{Z}/2) \subseteq H^4(R, \partial R; \mathbb{Z}/2)$$

is nonzero. By Edwards' Theorem, $\partial R'$ is a topological 5-manifold. Since $\Delta(\partial R')$ is zero, $\partial R'$ is actually homeomorphic to a P.L. 5-manifold V', that bounds a P.L. 6-manifold W'. Then let

$$N' := R' \cup U \cup W'$$

where U is the mapping cylinder of a homeomorphism between $\partial R'$ and $V' = \partial W'$. Since $\Delta(N')$ restricts to $\Delta(R')$, we have $\Delta(N') \neq 0$. Same as the original construction, we know that $\mathrm{Sq}^1(N') = w_1(N') \smile \Delta(N') \neq 0$

Proof of Theorem 4.2. Now we do hyperbolization on N' with some revises. Let (X^6, f) be the hyperbolized 6-simplex by Gromov's construction. Then let $R_1 := X^6 \Delta R'$, since $R' = R^5 \times S^1$ is still triangulated. Also, ∂R_1 is homeomorphic to a P.L. 5-manifold V (But the P.L. structure on V is incompatible with the triangulation of ∂R_1 as a subcomplex of R_1.) Let W be a P.L. manifold bounded by V, then apply relative hyperbolization to get an aspherical 6 manifold $R_2 = X\Delta(W, V)$ with boundary V. Then let U be the mapping cylinder of a homeomorphism $\partial R_1 \to V$. Let

$$N^6 = R_1 \cup U \cup R_2,$$

is an aspherical manifold as we want.

In the hyperbolization process, the links in polyhrdra are kept, indicating that $\Delta(R_1, \partial R_1)$ is same as $\Delta(R', \partial R') \neq 0$. By similar argument, $\Delta(N^6) \neq 0$, $\mathrm{Sq}^1(\Delta(N)) \neq 0$ and N^6 can not be triangulated.

Moreover, we can take $N^6 \times T^{n-6}$ as example of aspherical non-triangulable manifold for all $n \geqslant 6$.

□

4.3 Problems in Dimension 5

We cannot construct aspherical non-triangulable 5-manifold just following the above process, because $\partial R'$ in dimension 4 cannot admit P.L. structure.

However, if we could vary the definition of R^5 such that the homology manifold ∂R^5 becomes a P.L. 4-manifold after topological resolution, then we could keep on the hyperbolization process as above [5].

This problem remains open.

4.4 Whether Non-triangulable Manifolds Can Be Hyperbolized

There are two distinct flavors of non-triangulable construction, for $n = 4$ and $\geqslant 5$.

In dimension 4, our non-triangulable aspherical manifold N^4 is constructed by attaching the contractible F^4 to a hyperbolized 4-manifold "similar to" $Q(E_8)$ along the boundary P^3. However, Freedman's F^4, which is constructed using Casson handle to eliminate the π_1, contains infinite fractal structures, making it very hard to track the metric. That is to say, it's hard to tell if F^4 could admit CAT(0) metric. It would be a complicated work to follow rigorously how Casson handle performs in the topological infinite structure, to verify the possibility of tame metric on F^4. Otherwise, it may also possible to disprove it by contradiction, with help of the recent result:

Theorem 4.3 (Lytchak, Nagano, Stadler [13])**.** *Let X be a (globally)* CAT(0) *space which is a topological* 4*-manifold. Then X is homeomorphic to* $\mathbb{R}^4$.

we could first assume F^4 admit local CAT(0) metric, and attach it to other local CAT(0) space with boundary P^3 to get a complete local CAT(0) 4-manifold, whose universal cover is not $\mathbb{R}^4$, to get contradiction. However, when F^4 is treated with boundary P^3, the behavior is very complicated near the boundary.

However, in dimension $\geqslant 5$, it seems nothing to do with Freedman's work, because all we need is the algebraic result $\mathrm{Sq}^1(\Delta) \neq 0$. During this procedure, we did not use F^4 to topologically resolve the singularity, rather, we make use of the additional dimension to resolve the singularity similar to "double suspension theorem". So we cannot find the explicit location where the P.L. singularity live as in dimension 4. If we want to hyperbolize N^6 in Section 4.2, the difficulty comes from that we should find a CAT(0) metric on the topological mapping cylinder U between two non-equivalent P.L. structure, and the metric restricts to the boundary should be isometric to two original hyperbolized P.L. manifold. Now this approach is still of blank.

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致　谢

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